

The functional calculus of regular operators on Hilbert C^* -modules revisited

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Abstract

In [13], Woronowicz introduced a functional calculus for normal regular operators in a Hilbert C^* -module. In this paper, we have translated several concepts known in Hilbert space theory to the Hilbert C^* -module framework. We looked for instance into the functional calculus for strictly positive elements, the Fuglede Putnam theorem in Hilbert C^* -modules, commuting normal regular operators, ...

It appears that if we are a little bit careful, most of the Hilbert space results have their analogues in the Hilbert C^* -module case. An exception to this rule is of course the polar decomposition of a regular operator.

Introduction

Regular operators on Hilbert C^* -modules were studied in [1]. A nice extensive overview concerning Hilbert C^* -modules is given in [7]. Another standard reference for regular operators is [13].

There is a lot of interest for regular operators (or elements affiliated with a C^* -algebra) in the C^* -algebraic quantum group scene. There are several reasons for this :

- An interesting object associated to a locally compact group is the modular function which connects the left and right Haar measure. This modular function is a continuous group homomorphism from the group in the complex numbers.

In the quantum group case, the analogue of this modular function still exists, but it is now a strictly positive element affiliated with the C^* -algebra.

- Some important quantum groups are defined by generators and relations. In the compact case, this generators belong to the C^* -algebra. But in the non compact case, these generators can (and will be mostly) elements affiliated to the C^* -algebra (see e.g [13], [14]).

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Another application of regular operators can be found in the theory of so-called C^* -valued weights (see [6]).

The main aim of this paper is the translation of Hilbert space results to the Hilbert C^* -module framework in order to make the Hilbert C^* -module theory a little bit more flexible. Another part of this effort can be found in [5].

As a consequence, you will not find highly original results but only a collection of the most basic and useful properties. Most of the results are proven within the Hilbert C^* -module framework and do not make use of Hilbert space techniques.

The proofs of the results depend only on the results proven in [13] and [7].

The following topics are included in this paper :

1. Regular operators in Hilbert C^* -modules (preliminaries)
2. Left, right and middle multipliers
3. The functional calculus of normal elements (preliminaries)
4. A further development of the functional calculus
5. Invertible regular operators
6. The functional calculus revisited
7. The Fuglede-Putnam theorem for Hilbert C^* -modules
8. Natural left and right multipliers of a regular operator
9. C^* -algebras of adjointable operators on Hilbert C^* -modules
10. Representing Hilbert C^* -modules on Hilbert spaces
11. Commuting normal operators
12. Tensor products of Hilbert C^* -modules

We end this section with some conventions and notations.

If G is a set, we will denote the identity mapping on G by ι_G . If it is clear which set is under consideration, we will drop the subscript G . For any function f and any set G , we denote the restriction of f to G by the symbol f_G .

For any subset K of \mathbb{C} , we define $K_0 = K \setminus \{0\}$.

If E is a normed space, we denote the set of bounded operators by $\mathcal{B}(E)$.

Consider normed spaces E, F, G and linear mappings $S : E \rightarrow F$ and $T : F \rightarrow G$. If the composition TS is closable, we will always denote the closure by $S \cdot T$. We will however use this notation only in well controlled circumstances.

All Hilbert C^* -modules in these paper are right modules over the C^* -algebra in question. The inner valued product will always be linear in the first and adjointed linear in the second factor.

Consider Hilbert C^* -module E, F . Then we define $\mathcal{L}(E, F)$ as the set of adjointable operators from E into F . We will also use the notation $\mathcal{L}(E) = \mathcal{L}(E, E)$.

1 Regular operators on Hilbert C^* -modules

Regular operators on Hilbert C^* -modules were studied in [1]. Other standard references for regular operators are [13] and [7]. The results gathered in this first section come from [13] and [7].

Let us start of with the definition of the adjoint of a densely defined linear operator between Hilbert C^* -modules. The definition is completely same as the one for operators in Hilbert spaces but we have to use the C^* -algebra valued inner product.

Definition 1.1 Consider Hilbert C^* -modules E, F over a C^* -algebra A and let T be a densely defined A -linear mapping from within E into F . Then we define the mapping T^* from within F into E such that the domain of $D(T^*)$ is equal to

$$\{ v \in F \mid \text{There exists } w \in E \text{ such that } \langle T(u), v \rangle = \langle u, w \rangle \text{ for every } u \in D(T) \}$$

and such that $\langle T(u), v \rangle = \langle u, T^*(v) \rangle$ for every $v \in D(T^*)$ and $u \in D(T)$.

It is then easy to check that T^* is a closed A -linear mapping.

We can now introduce the definition of a regular operator (see page 96 of [7]).

Definition 1.2 Consider Hilbert C^* -modules E, F over a C^* -algebra A and let T be a densely defined A -linear mapping from within E into F such that T is closed, T^* is densely defined and $1 + T^*T$ has dense range. Then we call T a regular operator from within E into F .

Notation 1.3 Consider Hilbert C^* -modules E, F over a C^* -algebra A . Then we denote the set of regular operators from within E into F by $\mathcal{R}(E, F)$.

Remark 1.4 Consider a Hilbert C^* -module E over a C^* -algebra A . Then regular operators from within E into E are called regular operators in E . We will also use the notation $\mathcal{R}(E) = \mathcal{R}(E, E)$.

There is an alternative definition of a regular operator between Hilbert C^* -modules (see definition 1.1 of [13]). It is proven in [7] that both are equivalent.

Theorem 1.5 Consider Hilbert C^* -modules E, F over a C^* -algebra A and let T be a densely defined mapping from within E into F . Then T is regular \Leftrightarrow There exists an element $z \in \mathcal{L}(E, F)$ with $\|z\| \leq 1$ such that $D(T) = (1 - z^*z)^{\frac{1}{2}} E$ and $T((1 - z^*z)^{\frac{1}{2}} v) = z v$ for every $v \in E$.

Remark 1.6 Consider Hilbert C^* -modules E, F over a C^* -algebra A . Then we have the following properties concerning the element z in the definition above.

- Let T be an element in $\mathcal{R}(E, F)$. Then there exists a unique element $z \in \mathcal{L}(E, F)$ with $\|z\| \leq 1$ such that $D(T) = (1 - z^*z)^{\frac{1}{2}} E$ and $T((1 - z^*z)^{\frac{1}{2}} v) = z v$ for every $v \in E$. We use the notation $z_T = z$ and call z_T the z -transform of T .
- Consider two regular operators $S, T \in \mathcal{R}(E, F)$. Then $S = T$ if and only if $z_S = z_T$.
- Let z be an element in $\mathcal{L}(E, F)$ such that $\|z\| \leq 1$ and such that $(1 - z^*z)^{\frac{1}{2}} E$ is dense in E . Then there exists a unique element $T \in \mathcal{R}(E, F)$ such that $z_T = z$.

This z -transform turns out to be very useful in many proofs concerning regular operators. It allows to transfer problems concerning an unbounded regular operator to a bounded adjointable operator.

A first result in this respect connects the boundedness of T to a certain property of z_T :

Result 1.7 *Consider Hilbert C^* -modules E, F over a C^* -algebra A . Then the following holds :*

- *Consider $T \in \mathcal{L}(E, F)$. Then T is regular and $\|z_T\| < 1$.*
- *Consider $T \in \mathcal{R}(E, F)$. Then T belongs to $\mathcal{L}(E, F) \Leftrightarrow D(T) = E \Leftrightarrow T$ is bounded $\Leftrightarrow \|z_T\| < 1$.*

Remark 1.8 An important class of regular operators arises from C^* -algebras. In fact, Woronowicz looks in [13] only at this kind of regular operators but most of his proofs can be immediately copied to the case of regular operators between Hilbert C^* -modules.

Consider a C^* -algebra A and define E to be the Hilbert C^* -module over A such that $E = A$ as a right A -module and $\langle a, b \rangle = b^*a$ for every $a, b \in A$. Then the elements of $\mathcal{R}(E)$ are called elements affiliated with A . We write also $T \eta A$ instead of $T \in \mathcal{R}(E)$.

An important result, proven by Woronowicz in [13], states that a non-degenerate $*$ -homomorphism can be extended to the set of affiliated elements :

Theorem 1.9 *Consider a Hilbert C^* -module E over a C^* -algebra A . Let B be a C^* -algebra and π be a non-degenerate $*$ -homomorphism from B into $\mathcal{L}(E)$.*

Consider an element T affiliated with B . Then there exists a unique element $S \in \mathcal{R}(E)$ such that $z_S = \pi(z_T)$ and we define $S = \pi(T)$. We have moreover that $\pi(D(T))E$ is a core for $\pi(T)$ and $\pi(T)(\pi(b)v) = \pi(T(b))v$ for every $b \in D(T)$ and $v \in E$.

The last part of this theorem implies that $\pi(D)K$ is a core for $\pi(T)$ if D is a core for T and K is a dense subspace of E .

Remark 1.10 Suppose moreover that π is injective. Then the canonical extension of π to $M(B)$ is also injective. Using the z -transform, this implies immediately the following result.

Let S and T be two elements affiliated with B . Then $S = T$ if and only if $\pi(S) = \pi(T)$.

The following result can also be found in [13] (theorem 1.2). It follows easily using the z -transform.

Proposition 1.11 *Consider a Hilbert C^* -module E over a C^* -algebra A . Let B, C be two C^* -algebras. Consider a non-degenerate $*$ -homomorphism π from B into $M(C)$ and a non-degenerate $*$ -homomorphism θ from C into $\mathcal{L}(E)$. Then $(\theta\pi)(T) = \theta(\pi(T))$ for every $T \eta B$.*

Concerning the adjoint, we have the following key result (see theorem 1.4 of [13]).

Proposition 1.12 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and let T be an element in $\mathcal{R}(E, F)$. Then T^* is a regular operator from within F into E . We have moreover that $z_{T^*} = (z_T)^*$ and $T^{**} = T$.*

We have the usual definitions of selfadjointness, normality and positivity. Normality and selfadjointness were already considered in [13]. Positivity seems to be the logical definition.

Definition 1.13 Consider a Hilbert C^* -module E over a C^* -algebra A and let T be an element in $\mathcal{R}(E)$. Then we have the following definitions.

- We call T normal $\Leftrightarrow D(T) = D(T^*)$ and $\langle Tv, Tv \rangle = \langle T^*v, T^*v \rangle$ for every $v \in D(T)$.
- We call T selfadjoint $\Leftrightarrow T^* = T$
- We call T positive $\Leftrightarrow T$ is normal and $\langle Tv, v \rangle \geq 0$ for every $v \in D(T)$.

These definitions behave rather well with respect to the z -transforms (see equivalence 1.15 of [13]) :

Result 1.14 Consider a Hilbert C^* -module E over a C^* -algebra A and let T be an element in $\mathcal{R}(E)$. Then we have the following equivalencies.

- T is normal $\Leftrightarrow z_T$ is normal
- T is selfadjoint $\Leftrightarrow z_T$ is selfadjoint
- T is positive $\Leftrightarrow z_T$ is positive

Proof : The result concerning normality is proven in equivalence 1.15 of [13]. Using the fact that $z_{T^*} = (z_T)^*$, the result concerning the selfadjointness is trivial. So we turn to the positivity.

\Rightarrow Suppose that T is positive. Because T is normal, we have that z_T is normal.

Choose $v \in E$. We know that $(1 - z_T^* z_T)^{\frac{1}{2}} v$ belongs to $D(T)$ and $T((1 - z_T^* z_T)^{\frac{1}{2}} v) = z_T v$. Therefore, the positivity of T implies that the element $\langle z_T v, (1 - z_T^* z_T)^{\frac{1}{2}} v \rangle$ is positive.

Take $w \in E$, and replace v in the previous equality by $(1 - z_T^* z_T)^{\frac{1}{4}} w$. So we get that the element $\langle z_T (1 - z_T^* z_T)^{\frac{1}{4}} w, (1 - z_T^* z_T)^{\frac{3}{4}} w \rangle$ is positive.

By the normality of z_T , we have that

$$\begin{aligned} \langle z_T (1 - z_T^* z_T)^{\frac{1}{2}} w, (1 - z_T^* z_T)^{\frac{1}{2}} w \rangle &= \langle (1 - z_T^* z_T)^{\frac{1}{4}} z_T (1 - z_T^* z_T)^{\frac{1}{4}} w, (1 - z_T^* z_T)^{\frac{1}{2}} w \rangle \\ &= \langle z_T (1 - z_T^* z_T)^{\frac{1}{4}} w, (1 - z_T^* z_T)^{\frac{3}{4}} w \rangle \end{aligned}$$

So we see that the element $\langle z_T (1 - z_T^* z_T)^{\frac{1}{2}} w, (1 - z_T^* z_T)^{\frac{1}{2}} w \rangle$ is positive.

Because $(1 - z_T^* z_T)^{\frac{1}{2}} E$ is dense in E , this implies that $\langle z_T u, u \rangle$ is positive for every $u \in E$. By lemma 4.1 of [7], we get that z_T is positive.

\Leftarrow Suppose that z_T is positive. Because z_T is normal, T will be normal.

Choose $v \in D(T)$. Then there exists $w \in E$ such that $v = (1 - z_T^2)^{\frac{1}{2}} w$. This implies that

$$\langle Tv, v \rangle = \langle z_T w, (1 - z_T^2)^{\frac{1}{2}} w \rangle = \langle (1 - z_T^2)^{\frac{1}{4}} z_T w, (1 - z_T^2)^{\frac{1}{4}} w \rangle = \langle z_T (1 - z_T^2)^{\frac{1}{4}} w, (1 - z_T^2)^{\frac{1}{4}} w \rangle$$

which is positive by the positivity of z_T . ■

Remark 1.15 This result implies immediately that a positive regular operator is selfadjoint.

We will also need the following notion of strict positivity .

Definition 1.16 Consider a Hilbert C^* -module E over a C^* -algebra A and let T be an element in $\mathcal{R}(E)$. Then we call T strictly positive $\Leftrightarrow T$ is positive and T has dense range.

A strictly positive element is automatically injective. The reverse is not true. A counterexample can be easily constructed in the commutative case by taking a positive continuous function which is 0 in one point.

Strict positivity is intimately connected with invertibility of a regular operator which will be considered in section 5.

We have also the following results. The only difficult part of the proof of the first one is resolved by proposition 9.9 of [7]. The second one is the content of lemma 9.2 of [7].

Proposition 1.17 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and let T be an element in $\mathcal{R}(E, F)$. Then T^*T and TT^* are positive regular operators in E .*

Proposition 1.18 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and let T be an element in $\mathcal{R}(E, F)$. Then $D(T^*T)$ is a core for T .*

Result 1.19 *Consider a Hilbert C^* -module over a C^* -algebra A . Let B be a C^* -algebra and π a non-degenerate $*$ -homomorphism from B into $\mathcal{L}(E)$. Then we have for every element T affiliated with B the equalities*

$$\pi(T)^* = \pi(T^*)^* \quad \pi(T^*T) = \pi(T)^*\pi(T) \quad \pi(TT^*) = \pi(T)\pi(T)^*$$

Proof : We have that $z_{\pi(T^*)} = \pi(z_{T^*}) = \pi(z_T^*) = \pi(z_T)^* = z_{\pi(T)}^* = z_{\pi(T)^*}$. Hence $\pi(T^*) = \pi(T)^*$.

Take $x \in D(T^*T)$ and $v \in E$. Then $\pi(x)v$ belongs to $D(\pi(T^*T))$ and $\pi(T^*T)(\pi(x)v) = \pi((T^*T)(x))v$. Because x belongs to $D(T)$, we have that $\pi(x)v$ belongs to $D(\pi(T))$ and that $\pi(T)(\pi(x)v) = \pi(T(x))v$. Because $T(x)$ belongs to $D(T^*)$, this implies that $\pi(T)(\pi(x)v)$ belongs to $D(\pi(T^*))$ and that

$$\pi(T^*)(\pi(T)(\pi(x)v)) = \pi(T^*)(\pi(T(x))v) = \pi(T^*(T(x)))v = \pi((T^*T)(x))v = \pi(T^*T)(\pi(x)v) .$$

This implies that $\pi(x)v$ belongs to $D(\pi(T)^*\pi(T))$ and that $(\pi(T)^*\pi(T))(\pi(x)v) = \pi(T^*T)(\pi(x)v)$.

We know that $\pi(D(T^*T))E$ is a core for $\pi(T^*T)$. So we get that $\pi(T^*T) \subseteq \pi(T)^*\pi(T)$ by the closedness of $\pi(T)^*\pi(T)$.

Because both are selfadjoint (remember the first equality of this result), we get that $\pi(T^*T) = \pi(T)^*\pi(T)$. The other equality follows by symmetry. ■

Using the z -transform, the first three statements of the following proposition are easy to prove. The last one is then not so difficult to prove.

Proposition 1.20 *Consider a Hilbert C^* -module E over a C^* -algebra A . Let B be a C^* -algebra and π a non-degenerate $*$ -homomorphism from B into $\mathcal{L}(E)$. Suppose that T is an element affiliated with B . Then we have the following properties.*

1. *If T is normal, then $\pi(T)$ is normal.*
2. *If T is selfadjoint, then $\pi(T)$ is selfadjoint.*
3. *If T is positive, then $\pi(T)$ is positive.*
4. *If T is strictly positive, then $\pi(T)$ is strictly positive.*

If π is moreover injective, then the first three implications become equivalencies (use the z -transform). This is however not true for the last one!

2 Left, right and middle multipliers

This section serves merely to introduce some conventions, notations and basic results concerning certain compositions of adjointable and regular operators between Hilbert C^* -modules.

Let us start off with some definitions.

Terminology 2.1 Consider Hilbert C^* -modules E, F, G over a C^* -algebra A .

1. Consider $T \in \mathcal{R}(F, G)$ and $x \in \mathcal{L}(E, F)$. We call x a right multiplier of $T \Leftrightarrow Tx$ is bounded and $\overline{T x}$ belongs to $\mathcal{L}(E, G)$.
2. Consider $T \in \mathcal{R}(E, F)$ and $x \in \mathcal{L}(F, G)$. We call x a left multiplier of $T \Leftrightarrow xT$ is bounded and \overline{xT} belongs to $\mathcal{L}(E, G)$.

We will combine this with the following notations.

Notation 2.2 Consider Hilbert C^* -modules E, F, G over a C^* -algebra A .

1. Consider $T \in \mathcal{R}(F, G)$ and $x \in \mathcal{L}(E, F)$ such that x is a right multiplier of T . Then we define $T \cdot x = \overline{T x}$. So $T \cdot x$ belongs to $\mathcal{L}(E, G)$.
2. Consider $T \in \mathcal{R}(E, F)$ and $x \in \mathcal{L}(F, G)$ such that x is a left multiplier of T . Then we define $x \cdot T = \overline{xT}$. So $x \cdot T$ belongs to $\mathcal{L}(E, G)$.

Concerning right multipliers, we have the following useful characterization.

Result 2.3 Consider Hilbert C^* -modules E, F, G over a C^* -algebra A . Let T be an element in $\mathcal{R}(F, G)$ and x an element in $\mathcal{L}(E, F)$. Then

1. x is a right multiplier of $T \Leftrightarrow xE \subseteq D(T)$.
2. If x is a right multiplier of T , then $T \cdot x = Tx$.

Proof :

- Suppose that x is a right multiplier of T . Because $D(\overline{T x}) = E$, we have that $D(Tx)$ is dense in E .

Choose $v \in E$. Then there exists a sequence $(v_n)_{n=1}^\infty$ in $D(Tx)$ such that $(v_n)_{n=1}^\infty$ converges to v .

So $(xv_n)_{n=1}^\infty$ converges clearly to xv .

We have for every $n \in \mathbb{N}$ that xv_n belongs to $D(T)$ and that $T(xv_n) = (T \cdot x)v_n$. So we get that $(T(xv_n))_{n=1}^\infty$ converges to $(T \cdot x)v$. Hence, the closedness of T implies that xv belongs to $D(T)$ and $T(xv) = (T \cdot x)v$.

So we see that $xE \subseteq D(T)$ and that $Tx = T \cdot x$.

- Suppose that $xE \subseteq D(T)$. Then Tx is a closed linear operator from E into E so it must be bounded by the closed graph theorem. Then we have of course that $\overline{T x} = Tx$.

Because Tx is bounded, the operator $(Tx)^*$ is also bounded. We have also that $(Tx)^*$ is closed. It is not difficult to check that $x^*T^* \subseteq (Tx)^*$, so $(Tx)^*$ has a dense domain. Combining these three facts, we get that $D((Tx)^*) = E$. But this implies that Tx belongs to $\mathcal{L}(E, F)$. So x is by definition a right multiplier of T .

■

Another useful terminology is the following one :

Terminology 2.4 Consider Hilbert C^* -modules E, F, G, H over a C^* -algebra A . Let S be an element in $\mathcal{R}(E, F)$ and T an element in $\mathcal{R}(G, H)$. Consider $x \in \mathcal{L}(F, G)$. Then we call x a middle multiplier of S, T if and only if $x(\text{Ran } T) \subseteq D(S)$, SxT is bounded and \overline{SxT} belongs to $\mathcal{L}(E, H)$.

Notation 2.5 Consider Hilbert C^* -modules E, F, G, H over a C^* -algebra A . Let S be an element in $\mathcal{R}(E, F)$ and T an element in $\mathcal{R}(G, H)$. Consider $x \in \mathcal{L}(F, G)$ such that x is a middle multiplier of S, T . Then we define $S \cdot x \cdot T = \overline{SxT}$. So $S \cdot x \cdot T$ belongs to $\mathcal{L}(E, H)$.

Remark 2.6 Consider Hilbert C^* -modules E, F, G over a C^* -algebra A and let S be an element in $\mathcal{R}(F, G)$. Consider an element x in $\mathcal{L}(E, F)$. Then we have clearly the following properties :

- x is a right multiplier of $S \Leftrightarrow x$ is a middle multiplier of $S, 1$.
- If x is a right multiplier of S , then $S \cdot x = S \cdot x \cdot 1$.

A similar remark applies of course also for left multipliers.

Concerning the adjoint operation, we have the following expected result.

Result 2.7 Consider Hilbert C^* -modules E, F, G, H over a C^* -algebra A . Let S be an element in $\mathcal{R}(E, F)$ and T an element in $\mathcal{R}(G, H)$. Consider $x \in \mathcal{L}(F, G)$. Then we have the following properties.

- x is a middle multiplier of $S, T \Leftrightarrow x^*$ is a middle multiplier of T^*, S^*
- If x is a middle multiplier of S, T , then $(S \cdot x \cdot T)^* = T^* \cdot x^* \cdot S^*$.

Proof : Suppose that x is a middle multiplier of S, T .

Take $v \in D(S^*)$. Then we have for every $w \in D(T)$ that

$$\langle x^* S^*(v), T(w) \rangle = \langle S^*(v), xT(w) \rangle = \langle v, S(xT(w)) \rangle = \langle v, (S \cdot x \cdot T)w \rangle = \langle (S \cdot x \cdot T)^* v, w \rangle$$

which implies that $x^* S^*(v)$ belongs to $D(T^*)$ and that $T^*(x^* S^*(v)) = (S \cdot x \cdot T)^* v$.

So we get by definition that x^* is a middle multiplier of T^*, S^* and $T^* \cdot x^* \cdot S^* = (S \cdot x \cdot T)^*$.

If x^* is a middle multiplier of T^*, S^* , we get in a similar way that x is a middle multiplier of S, T . ■

A special case hereof is the following corollary (see also remark 2.6).

Corollary 2.8 Consider Hilbert C^* -modules E, F, G over a C^* -algebra A and let T be an element in $\mathcal{R}(E, F)$ and x an element in $\mathcal{L}(F, G)$. Then we have the following properties.

- x is a left multiplier of $T \Leftrightarrow x^*$ is a right multiplier of T^*
- If x is a left multiplier of T , then $(x \cdot T)^* = T^* \cdot x^*$.

Remark 2.9 We have all kinds of multiplication properties (each of which is easy to prove). For instance, we have the following one.

Consider Hilbert C^* -modules E, F, G, H over a C^* -algebra A . Let $T \in \mathcal{R}(E, F)$, $y \in \mathcal{L}(F, G)$ and $x \in \mathcal{L}(G, H)$ such that y is a left multiplier of T . Then xy is a left multiplier of T and $(xy) \cdot T = x(y \cdot T)$.

Proposition 2.10 Consider a Hilbert C^* -module E over a C^* -algebra A . Let B be a C^* -algebra and π a non-degenerate $*$ -homomorphism from B into $\mathcal{L}(E)$. Consider $S, T \in \mathcal{R}(B)$ and $x \in \mathcal{L}(B)$ such that x is a middle multiplier of S, T . Then $\pi(x)$ is a middle multiplier of S, T and $\pi(S \cdot x \cdot T) = \pi(S) \cdot \pi(x) \cdot \pi(T)$.

Proof : Choose $a \in D(T)$ and $v \in E$. Then $\pi(a)v \in D(\pi(T))$ and $\pi(T)(\pi(a)v) = \pi(T(a))v$. Because x is a middle multiplier of S, T , we have that $xT(a)$ belongs to $D(S)$ and $S(xT(a)) = (S \cdot x \cdot T)a$. This implies that $\pi(xT(a))v \in D(\pi(S))$ and

$$\pi(S)(\pi(xT(a))v) = \pi(S(xT(a)))v = \pi((S \cdot x \cdot T)a)v = \pi(S \cdot x \cdot T)\pi(a)v$$

So we get that $\pi(x)(\pi(T)(\pi(a)v))$ belongs to $D(\pi(S))$ and $\pi(S)(\pi(x)(\pi(T)(\pi(a)v))) = \pi(S \cdot x \cdot T)\pi(a)v$. We know that $\pi(D(T))E$ is a core for $\pi(T)$. Combining this with the closedness of $\pi(S)$, the last result implies easily for every $w \in D(\pi(T))$ that $\pi(x)(\pi(T)w) \in D(\pi(S))$ and $\pi(S)(\pi(x)(\pi(T)w)) = \pi(S \cdot x \cdot T)w$.

So we have also that $\pi(S)\pi(x)\pi(T)$ is bounded and $\overline{\pi(S)\pi(x)\pi(T)} = \pi(S \cdot x \cdot T)$ which belongs to $\mathcal{L}(E)$. Consequently $\pi(x)$ is a middle multiplier of $\pi(S), \pi(T)$ and $\pi(S) \cdot \pi(x) \cdot \pi(T) = \pi(S \cdot x \cdot T)$. ■

Corollary 2.11 Consider a Hilbert C^* -module E over a C^* -algebra A . Let B be a C^* -algebra and π a non-degenerate $*$ -homomorphism from B into $\mathcal{L}(E)$. Consider moreover $T \in \mathcal{R}(B)$ and $x \in \mathcal{L}(E)$.

- If x is a right multiplier of T , then $\pi(x)$ is a right multiplier of $\pi(T)$ and $\pi(T) \cdot \pi(x) = \pi(T \cdot x)$.
- If x is a left multiplier of T , then $\pi(x)$ is a left multiplier of $\pi(T)$ and $\pi(x) \cdot \pi(T) = \pi(x \cdot T)$.

3 The functional calculus of normal elements

In [13], Woronowicz introduced the functional calculus of normal regular operators. We will give an overview of the most important results.

The main result is the content of the next theorem (see theorem 1.5 of [13]).

Theorem 3.1 Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Then there exists a unique non-degenerate $*$ -homomorphism φ_T from $C_0(\mathbb{C})$ into $\mathcal{L}(E)$ such that $\varphi_T(\iota_{\mathbb{C}}) = T$.

If T belongs to $\mathcal{L}(E)$, we have already a notion of a functional calculus and spectrum for T (looked upon as an element in the C^* -algebra $\mathcal{L}(E)$).

Lemma 3.2 Consider a Hilbert C^* -module E over a C^* -algebra A . Let T be a normal element in $\mathcal{L}(E)$. Then we have the following properties.

- We have for every $f \in C(\mathbb{C})$ that $\varphi_T(f) = f(T)$.

- The spectrum $\sigma(T)$ is equal to the set $\{c \in \mathbb{C} \mid f(c) = 0 \text{ for all } f \in \ker \varphi_T\}$

Proof :

- Define the mapping π from $C_0(\mathbb{C})$ into $\mathcal{L}(E)$ such that $\pi(f) = f_{\sigma(T)}(T)$ for every $f \in C_0(\mathbb{C})$. As a composition of non-degenerate *-homomorphisms, π is non-degenerate.

Using proposition 1.11, it is not so difficult to check that $\pi(f) = f_{\sigma(T)}(T)$ for every $f \in C(\sigma(T))$.

We have in particular that $\pi(\iota_{\mathbb{C}}) = \iota_{\sigma(T)}(T) = T$. So $\varphi_T = \pi$.

- The inclusion \subseteq follows immediately from the previous property. The inclusion \supseteq requires the lemma of Urysohn ($\sigma(T)$ is compact!).

■

Woronowicz introduced the following consistent definition.

Definition 3.3 Consider a Hilbert C^* -module E over a C^* -algebra A . Let T be a normal element in $\mathcal{R}(E)$. Then the spectrum of T is defined to be set

$$\sigma(T) = \{c \in \mathbb{C} \mid f(c) = 0 \text{ for all } f \in \ker \varphi_T\}.$$

It is clear that $\sigma(T)$ is a closed subset of \mathbb{C} . Because φ_T is non-degenerate, $\varphi_T \neq 0$ so $\ker \varphi_T \neq C_0(\mathbb{C})$. We have also that $C_0(\sigma(T)) \cong C_0(\mathbb{C})/\ker \varphi_T$, which implies that $\sigma(T) \neq \emptyset$.

Using this spectrum, we have the following theorem (theorem 1.6 of [13]).

Theorem 3.4 Consider a Hilbert C^* -module E over a C^* -algebra A . Let T be a normal element in $\mathcal{R}(E)$. Then there exists a unique non-degenerate injective *-homomorphism ψ_T from $C_0(\sigma(T))$ into $\mathcal{L}(E)$ such that $\psi_T(\iota_{\sigma(T)}) = T$.

If T belongs to $\mathcal{L}(E)$, we have immediately that $\psi_T(f) = f(T)$ for every $f \in C(\sigma(T))$. So we get in this case for every continuous function f from within \mathbb{C} into \mathbb{C} which is defined on $\sigma(T)$ that $f(T) = f_{\sigma(T)}(T) = \psi_T(f_{\sigma(T)})$. This remark justifies the following notation.

Notation 3.5 Consider a Hilbert C^* -module E over a C^* -algebra A . Let T be a normal element in $\mathcal{R}(E)$. For every continuous function f from within \mathbb{C} into \mathbb{C} which is defined on $\sigma(T)$, we define the element $f(T) = \psi_T(f_{\sigma(T)})$, so $f(T)$ is a normal element in $\mathcal{R}(E)$.

If f is bounded on $\sigma(T)$, we see that $f(T)$ belongs to $\mathcal{L}(E)$.

We have for every $f \in C(\sigma(T))$, that $f(T) = \psi_T(f)$. This implies immediately the usual functional calculus rules for elements in $C_b(\sigma(T))$.

Lemma 3.6 Consider a Hilbert C^* -module E over a C^* -algebra A . Let T be a normal element in $\mathcal{R}(E)$. We have for every $f \in C(\mathbb{C})$ that $\varphi_T(f) = f(T)$.

Proof : Define the non-degenerate *-homomorphism π from $C_0(\mathbb{C})$ into $C_b(\sigma(T))$ such that $\pi(f) = f_{\sigma(T)}$ for every $f \in C_0(\mathbb{C})$. Then $\pi(f) = f_{\sigma(T)}$ for every $f \in C(\mathbb{C})$.

We have that $\psi_T \pi$ is a non-degenerate *-homomorphism from $C_0(\mathbb{C})$ into $\mathcal{L}(E)$ such that $(\psi_T \pi)(\iota_{\mathbb{C}}) = \psi_T(\iota_{\sigma(T)}) = T$. But this implies that $\psi_T \pi = \varphi_T$, so we have for every $f \in C(\mathbb{C})$ that

$$\varphi_T(f) = \psi_T(\pi(f)) = \psi_T(f_{\sigma(T)}) = f_{\sigma(T)}(T) = f(T).$$

■

The spectrum of T and z_T are intimately connected (See the remarks at the bottom of page 408 of [13]).

Result 3.7 Consider a Hilbert C^* -module E over a C^* -algebra A . Let T be a normal element in $\mathcal{R}(E)$. Then

1. $\sigma(T) = \left\{ \frac{c}{(1-|c|^2)^{\frac{1}{2}}} \mid c \in \sigma(z_T) \text{ with } |c| < 1 \right\}$.
2. $\sigma(z_T) = \text{the closure of } \left\{ \frac{c}{(1+|c|^2)^{\frac{1}{2}}} \mid c \in \sigma(T) \right\}$.

Proof : The proof of this result depends on the proof of theorem 1.5 of [13]. Call D the closed unit disk in the complex plane, D^0 its interior and ∂D its boundary. Define the function J from \mathbb{C} into \mathbb{C} such that $J(c) = \frac{c}{(1+|c|^2)^{\frac{1}{2}}}$ for every $c \in \mathbb{C}$. Then J is a homeomorphism from \mathbb{C} to D^0 such that $J^{-1}(c) = \frac{c}{(1-|c|^2)^{\frac{1}{2}}}$ for every c in D^0 .

Some reflection allows us to conclude that

$$C_0(\mathbb{C}) = \{ f \circ J \mid f \in C(D) \text{ such that } f = 0 \text{ on } \partial D \} . \quad (\text{a})$$

Looking at the proof of theorem 1.5 of [13] (equality 1.19), we know that $\varphi_T(f \circ J) = f(z_T)$ for every $f \in C(D)$ (b)

Because $\sigma(z_T) \subseteq D$, we have, similar to the equation in lemma 3.2, that

$$\sigma(z_T) = \{ d \in D \mid f(d) = 0 \text{ for all } f \in C(D) \text{ such that } f(z_T) = 0 \} \quad (\text{c})$$

which implies that the set $\sigma(z_T) \cap D^0$ is equal to the set

$$\{ d \in D^0 \mid f(d) = 0 \text{ for all } f \in C(D) \text{ such that } f(z_T) = 0 \text{ and } f = 0 \text{ on } \partial D \} .$$

For inclusion \supseteq , use the fact that for every $d \in D^0$ there exists an element $g \in C(D)$ such that $g(d) = 1$ and $g = 0$ on ∂D .

Combining this equality, the definition of $\sigma(T)$ and equalities (a) and (b), it is now easy to prove that $J(\sigma(T)) = \sigma(z_T) \cap D^0$.

The previous equation implies that $J(\sigma(T)) \subseteq \sigma(z_T)$, so $\overline{J(\sigma(T))} \subseteq \sigma(z_T)$.

Choose $d \in \sigma(z_T)$. Take $f \in C(D)$ such that $f = 0$ on $J(\sigma(T))$. Then $f \circ J = 0$ on $\sigma(T)$, which implies that $\varphi_T(f \circ J) = 0$ by the lemma before this result. So $f(z_T) = \varphi_T(f \circ J) = 0$. This implies that $f(d) = 0$ by (c). So the lemma of Urysohn implies that d belongs to $\overline{J(\sigma(T))}$.

Consequently we get that $\sigma(z_T) = \overline{J(\sigma(T))}$. ■

Corollary 3.8 Consider a Hilbert C^* -module E over a C^* -algebra A . Let T be a normal element in $\mathcal{R}(E)$. Then

1. T is selfadjoint $\Leftrightarrow \sigma(T) \subseteq \mathbb{R}$.
2. T is positive $\Leftrightarrow \sigma(T) \subset \mathbb{R}^+$.

Proof : If T is selfadjoint, we know that z_T is selfadjoint which implies that $\sigma(z_T) \subseteq \mathbb{R}$, therefore $\sigma(T) \subseteq \mathbb{R}$. If $\sigma(T) \subseteq \mathbb{R}$, then $\iota_{\sigma(T)}$ is real valued which implies that $T = \psi_T(\iota_{\sigma(T)})$ is selfadjoint.

The result about positivity is proven in a similar way. ■

So we see that the usual results about selfadjointness and positivity remain true.

Also, the following result holds (remarks concerning equality 1.21 of [13])

Proposition 3.9 *Consider a Hilbert C^* -module E over a C^* -algebra A . Let B be a C^* -algebra and π a non-degenerate $*$ -homomorphism from B into $\mathcal{L}(E)$. Let T be a normal element affiliated with B . Then $\pi(T)$ is a normal element in $\mathcal{R}(E)$ such that $\sigma(\pi(T)) \subseteq \sigma(T)$. We have moreover that $f(\pi(T)) = \pi(f(T))$ for every continuous function f from within \mathbb{C} into \mathbb{C} which is defined on $\sigma(T)$.*

If π is injective, then we have even that $\sigma(\pi(T)) = \sigma(T)$. This follows easily from result 3.7 because in this case $\bar{\pi}$ is injective which implies that $\sigma(z_{\pi(T)}) = \sigma(\pi(z_T)) = \sigma(z_T)$.

The spectrum of a continuous function on a locally compact space is the closure of its image. Combining this with the previous remark gives

Result 3.10 *Consider a Hilbert C^* -module E over a C^* -algebra A . Let T be a normal element in $\mathcal{R}(E)$. Let f be a continuous function from within \mathbb{C} into \mathbb{C} which is defined on $\sigma(T)$. Then $\sigma(f(T)) = \overline{f(\sigma(T))}$.*

The following result is a special case of proposition 3.9 with π equal to ψ_T .

Proposition 3.11 *Consider a Hilbert C^* -module E over a C^* -algebra A . Let T be a normal element in $\mathcal{R}(E)$. Let f be a continuous function from within \mathbb{C} into \mathbb{C} defined on $\sigma(T)$ and g a continuous function from within \mathbb{C} into \mathbb{C} defined on $\sigma(f(T))$. Then $g \circ f$ is defined on $\sigma(T)$ and $(g \circ f)(T) = g(f(T))$.*

We end this section with a familiar alternative condition for normality.

Proposition 3.12 *Consider a Hilbert C^* -module E over a C^* -algebra A . Let T be an element in $\mathcal{R}(E)$. Then T is normal $\Leftrightarrow T^*T = TT^*$.*

Proof :

- Suppose that T is normal. Define the function f from $\sigma(T)$ into \mathbb{C} such that $f(c) = c$ for every $c \in \sigma(T)$. So $\psi_T(f) = T$.

It is clear that $f^*f = ff^*$, so result 1.19 implies that

$$T^*T = \psi_T(f)^*\psi_T(f) = \psi_T(f^*f) = \psi_T(ff^*) = \psi_T(f)\psi_T(f^*) = TT^*.$$

- Suppose that $T^*T = TT^*$. By proposition 1.18, we have in this case that $D(T^*T)$ is a core for both T^*T and TT^* .

Because $T^*T = TT^*$, we have moreover that $\langle Tv, Tv \rangle = \langle T^*v, T^*v \rangle$ for every $v \in D(T^*T)$.

Using the fact that $D(T^*T)$ is a core for both T^*T and TT^* , we get now easily that $D(T) = D(T^*)$.

It is now also easy to prove that $\langle Tv, Tv \rangle = \langle T^*v, T^*v \rangle$ for every $v \in D(T)$. ■

4 A further development of the functional calculus

In this section we are going to prove some results which makes it possible to develop the functional calculus for normal elements a little bit further than in the previous section.

Essentially, we are going to say something about calculation rules involving (unbounded) continuous functions of normal affiliated elements.

We will work a little bit more general than the situation described above because this will be useful to us in later applications.

Except for the last result in this section, we will fix a Hilbert C^* -module E over a C^* -algebra A , a locally compact space X and a non-degenerate $*$ -homomorphism π from $C_0(X)$ into $\mathcal{L}(E)$.

If f belongs to $C(X)$, then f is affiliated to $C_0(X)$ so we get a normal element $\pi(f)$ in $\mathcal{R}(E)$.

We have the following easy lemma.

Lemma 4.1 *Consider $f \in C(X)$.*

1. *We have for every $g \in K(X)$ that g belongs to $D(f)$.*
2. *$K(X)$ is a core for f .*

Proof : By [13], we know that $D(f) = \{ g \in C_0(X) \mid fg \in C_0(X) \}$ and $f(g) = fg$ for every $g \in D(f)$. We have for every $g \in K(X)$ that fg belongs to $K(X)$, so g belongs to $D(f)$ by the previous remark. Take an approximate unit $(e_k)_{k \in K}$ for $C_0(X)$ in $K(X)$.

Choose $h \in D(f)$. We have for every $k \in K$ that he_k belongs to $K(X)$ and $f(he_k) = fhe_k$. Because h and fh belong to $C_0(X)$ we see that $(he_k)_{k \in K}$ converges to h and that $(f(he_k))_{k \in K}$ converges to $fh = f(h)$. ■

By theorem 1.9 and the remark after it, this implies immediately the following lemma.

Lemma 4.2 *Consider $f \in C(X)$. Then the following properties hold.*

- *We have for every $g \in K(X)$ and $v \in E$ that $\pi(g)v$ belongs to $D(\pi(f))$ and $\pi(f)(\pi(g)v) = \pi(fg)v$.*
- *The space $\pi(K(X))E$ is a core for $\pi(f)$.*

Lemma 4.3 *Consider $f \in C(X)$, $g \in K(X)$. Then $\pi(g)$ is a left and right multiplier of $\pi(f)$ and*

$$\pi(g)\pi(f) \subseteq \pi(f)\pi(g) = \pi(fg)$$

Proof : The previous lemma implies by result 2.3 that $\pi(g)$ is a right multiplier of $\pi(f)$ and that $\pi(f) \cdot \pi(g) = \pi(fg)$.

The previous lemma implies of course also that $\pi(\bar{g})$ is a right multiplier of $\pi(\bar{f})$ and $\pi(\bar{f}) \cdot \pi(\bar{g}) = \pi(\overline{fg})$. Hence, corollary 2.8 implies that $\pi(g)$ is a left multiplier of $\pi(f)$ and $\pi(g) \cdot \pi(f) = \pi(fg)$. ■

This lemma implies, as usual, the following result

Result 4.4 *Consider a net $(e_k)_{k \in K}$ in $K(X)$ which is bounded and converges strictly to 1. Let f be an element in $C(X)$. Then we have the following properties.*

1. *We have for every $v \in D(\pi(f))$ that the net $(\pi(fe_k)v)_{k \in K}$ converges to $\pi(f)v$.*
2. *Let $v \in E$. Then v belongs to $D(\pi(f)) \Leftrightarrow$ the net $(\pi(fe_k)v)_{k \in K}$ is convergent in E .*

Like in the Hilbert space case (see e.g. [10]), we will use this truncating net $(\pi(e_k))_{k \in K}$ to get more detail into the functional calculus of unbounded continuous functions. This result will be freely used in the sequel.

Result 4.5 Consider $f_1, \dots, f_n \in C(X)$. Then $\pi(f_1) \dots \pi(f_n)$ is closable and we have the equality

$$\overline{\pi(f_1) \dots \pi(f_n)} = \pi(f_1 \dots f_n)$$

Proof : Take an approximate unit $(e_k)_{k \in K}$ for $C_0(X)$ in $K(X)$.

Using lemma 4.3 and induction, it is easy to prove that

$$\pi(e_k)\pi(f_1) \dots \pi(f_n) \subseteq \pi(f_1) \dots \pi(f_n)\pi(e_k) = \pi(f_1 \dots f_n e_k) \in \mathcal{L}(E)$$

for $k \in K$.

- Choose $v \in D(\pi(f_1) \dots \pi(f_n))$. By lemma 4.3 and the relation above, we have for every $k \in K$ that $\pi(e_k)v \in D(\pi(f_1 \dots f_n))$ and that

$$\pi(f_1 \dots f_n) \pi(e_k)v = \pi(e_k) (\pi(f_1) \dots \pi(f_n)) v$$

This implies that the net $(\pi(f_1 \dots f_n) \pi(e_k)v)_{k \in K}$ converges to $(\pi(f_1) \dots \pi(f_n)) v$. Combining this with the closedness of $\pi(f_1 \dots f_n)$ and the fact that $(\pi(e_k)v)_{k \in K}$ converges to v , we see that $v \in D(\pi(f_1 \dots f_n))$ and that $\pi(f_1 \dots f_n) v = (\pi(f_1) \dots \pi(f_n)) v$.

So we see that $\overline{\pi(f_1) \dots \pi(f_n)} \subseteq \pi(f_1 \dots f_n)$ which implies that $\pi(f_1) \dots \pi(f_n)$ is closable and that $\overline{\pi(f_1) \dots \pi(f_n)} = \pi(f_1 \dots f_n)$.

- Choose $v \in D(\pi(f_1 \dots f_n))$. By the relation above, we have for every $k \in K$ that $\pi(e_k)v \in D(\pi(f_1) \dots \pi(f_n))$.

We know by result 4.4 also that the net $(\pi(f_1 \dots f_n) \pi(e_k)v)_{k \in K}$ converges to $\pi(f_1 \dots f_n) v$. We have of course also that the net $(\pi(e_k)v)_{k \in K}$ converges to v .

So we see that $D(\pi(f_1) \dots \pi(f_n))$ is a core for $\pi(f_1 \dots f_n)$. ■

We will need the following notation later on.

Notation 4.6 Consider $f_1, \dots, f_n \in C(X)$. Then we define the element $\pi(f_1) \cdot \dots \cdot \pi(f_n)$ as the closure of $\pi(f_1) \dots \pi(f_n)$. So $\pi(f_1) \cdot \dots \cdot \pi(f_n) = \overline{\pi(f_1) \dots \pi(f_n)}$.

Result 4.7 Consider $f, g \in C(X)$. Then $D(\pi(f)\pi(g)) = D(\pi(g)) \cap D(\pi(fg))$.

Proof : Take an approximate unit $(e_k)_{k \in K}$ for $C_0(X)$ in $K(X)$.

Because $\pi(f)\pi(g) \subseteq \pi(fg)$, we have immediately that $D(\pi(f)\pi(g))$ is a subset of $D(\pi(fg)) \cap D(\pi(g))$.

Choose $v \in D(\pi(fg)) \cap D(\pi(g))$. Because $v \in D(\pi(fg))$, we have that the net $(\pi(fge_k)v)_{k \in K}$ converges to $\pi(fg)v$.

Lemma 4.3 implies that $\pi(fe_k)(\pi(g)v) = \pi(fge_k)v$ for every $k \in K$. Consequently, the net

$(\pi(fe_k)(\pi(g)v))_{k \in K}$ is convergent in E , implying that $\pi(g)v$ belongs to $D(\pi(f))$.

Hence, $D(\pi(f)\pi(g)) = D(\pi(fg)) \cap D(\pi(g))$. ■

Corollary 4.8 Consider $f \in C(X)$ and $g \in C_b(X)$. Then $\pi(f)\pi(g) = \pi(fg)$.

Corollary 4.9 Consider $f \in C(X)$ and $g \in C_b(X)$ such that $fg \in C_b(X)$. Then $\pi(g)$ is a left and a right multiplier of $\pi(f)$ and

$$\pi(g)\pi(f) \subseteq \pi(f)\pi(g) = \pi(fg)$$

The proofs of the following results are based on the same principle as the proof above. Therefore we leave them out.

Result 4.10 Consider $f_1, \dots, f_n \in C(X)$. Then $\pi(f_1) + \dots + \pi(f_n)$ is closable and we have the equality

$$\overline{\pi(f_1) + \dots + \pi(f_n)} = \pi(f_1 + \dots + f_n)$$

Corollary 4.11 Consider $f, g \in C(X)$ such that f or g is bounded. Then $\pi(f) + \pi(g) = \pi(f + g)$.

Result 4.12 Consider $f \in C(X)$ and $c \in \mathbb{C}_0$. Then $\pi(cf) = c\pi(f)$.

The next result is true because taking the adjoint commutes with every non-degenerate *-homomorphism.

Result 4.13 Consider $f \in C(X)$. Then $\pi(f)^* = \pi(\bar{f})$.

Result 4.14 Let $f \in C(X)$. Then $\pi(f)^*\pi(f) = \pi(f)\pi(f)^* = \pi(|f|^2)$.

Proof : We know that $\pi(f)^*\pi(f)$ is closed. Hence,

$$\pi(|f|^2) = \pi(\bar{f}f) = \overline{\pi(\bar{f})\pi(f)} = \overline{\pi(f)^*\pi(f)} = \pi(f)^*\pi(f) .$$

Similarly, we have that $\pi(|f|^2) = \pi(f)\pi(f)^*$. ■

The next three results will be very useful to us in later sections.

Lemma 4.15 Consider $f, g \in C(X)$ such that there exists a positive number r and a compact subset M of X such that $|f(x)| \leq r|g(x)|$ for every $x \in X \setminus M$. Then $D(\pi(g)) \subset D(\pi(f))$.

Proof : Take an approximate unit $(e_k)_{k \in K}$ for $C_0(X)$ in $K(X)$ such that $e_k = 1$ on M for every $k \in K$ (This is possible because of the lemma of Urysohn).

Choose $v \in D(\pi(g))$.

Take $k, l \in K$. Because $e_k = 1$ on M , $e_l = 1$ on M and because we have that $|f(c)| \leq r|g(c)|$ for every $c \in X \setminus M$, we see that

$$|fe_k - fe_l| = |f||e_k - e_l| \leq r|g||e_k - e_l| = |ge_k - ge_l| .$$

So we have that

$$\begin{aligned} \langle \pi(fe_k)v - \pi(fe_l)v, \pi(fe_k)v - \pi(fe_l)v \rangle &= \langle (\pi(fe_k) - \pi(fe_l))^*(\pi(fe_k) - \pi(fe_l))v, v \rangle \\ &= \langle \pi(|fe_k - fe_l|^2)v, v \rangle \\ &\leq r^2 \langle \pi(|ge_k - ge_l|^2)v, v \rangle \\ &= r^2 \langle (\pi(ge_k) - \pi(ge_l))^*(\pi(ge_k) - \pi(ge_l))v, v \rangle \\ &= r^2 \langle \pi(ge_k)v - \pi(ge_l)v, \pi(ge_k)v - \pi(ge_l)v \rangle \end{aligned}$$

which implies that $\|\pi(fe_k)v - \pi(fe_l)v\| \leq r \|\pi(ge_k)v - \pi(ge_l)v\|$.

Because v belongs to $D(\pi(g))$, we know that $(\pi(ge_k)v)_{k \in K}$ is convergent in E . Therefore, the previous estimation guarantees that $(\pi(fe_k)v)_{k \in K}$ is also convergent in E , which in turn implies that v belongs to $D(\pi(f))$. ■

Lemma 4.16 *Consider $f, g \in C(X)$ such that there exists a positive number r such that $|f(x)| \leq r |g(x)|$ for every $x \in X$. Then $\langle \pi(f)v, \pi(f)v \rangle \leq r^2 \langle \pi(g)v, \pi(g)v \rangle$ for every $v \in D(\pi(g))$.*

Proof : Remember from the previous lemma that $D(\pi(g)) \subseteq D(\pi(f))$.

Take an approximate unit $(e_k)_{k \in K}$ for $C_0(X)$ in $K(X)$.

Then $(\pi(fe_k)v)_{k \in K}$ converges to $\pi(f)v$ and $(\pi(ge_k)v)_{k \in K}$ converges to $\pi(g)v$.

We have for every $k \in K$ that

$$\langle \pi(fe_k)v, \pi(fe_k)v \rangle = \langle \pi(|fe_k|^2)v, v \rangle \leq r^2 \langle \pi(|ge_k|^2)v, v \rangle = r^2 \langle \pi(ge_k)v, \pi(ge_k)v \rangle$$

So the lemma follows now immediately. ■

Lemma 4.17 *Consider $f \in C(X)$, M a compact subset of X and r a positive number such that $|f(x)| > r$ for every $x \in M$. Then there exist an element $g \in K(X)$ such that $g = 1$ on M , $0 \leq g \leq 1$ and $r^2 \langle \pi(g)v, \pi(g)v \rangle \leq \langle \pi(f)v, \pi(f)v \rangle$ for every $v \in D(\pi(f))$.*

Proof : If $M = \emptyset$ or $r = 0$, the lemma is trivially true. So suppose that $M \neq \emptyset$ and $r \neq 0$.

Define for every $x \in M$ the set $O_x = \{y \in X \mid |f(y)| > r\}$, then O_x is an open neighbourhood of x . By the compactness of M , there exist $x_1, \dots, x_n \in M$ such that $M \subseteq O_{x_1} \cup \dots \cup O_{x_n}$.

Put $U = O_{x_1} \cup \dots \cup O_{x_n}$. Then U is an open subset of X such that $M \subseteq U$ and $|f(y)| \geq r$ for every $y \in U$.

By the lemma of Urysohn, there exist an element $g \in K(X)$ such that $g = 1$ on M , $g = 0$ on $X \setminus U$ and $0 \leq g \leq 1$. Then it is clear that $|f| \geq rg$. The previous lemma now guarantees that $r^2 \langle \pi(g)v, \pi(g)v \rangle \leq \langle \pi(f)v, \pi(f)v \rangle$ for every $v \in D(\pi(f))$. ■

We end this section with a result of which the proof is easy in the Hilbert space case because then $\overline{\text{Ran } T^*} = (\ker T)^\perp$. The bounded case has already been proven in proposition 3.7 of [7]. We will use in fact this result in the proof of the next proposition.

Proposition 4.18 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and an element $T \in \mathcal{R}(E, F)$. Then we have that $\overline{\text{Ran } T^*T} = \overline{\text{Ran } T^*}$ and $\ker T^*T = \ker T$.*

Proof : Take a bounded net $(e_i)_{i \in I}$ in $K(\mathbb{C})$ such that $(e_i)_{i \in I}$ converges strictly to 1.

Take $j \in J$. Then $e_j(T^*T)E \subseteq D(T^*T)$ by lemma 4.3, so we have certainly that $e_j(T^*T)E \subseteq D(T)$.

Therefore is $e_j(T^*T)$ a right multiplier of T . So we have the element $T e_j(T^*T) \in \mathcal{L}(E, F)$.

Hence, proposition 3.7 of [7] implies that

$$\overline{\text{Ran } (T e_j(T^*T))^*} = \overline{\text{Ran } (T e_j(T^*T))^* (T e_j(T^*T))}$$

It is clear that $e_j(T^*T)T^* \subseteq (T e_j(T^*T))^*$.

This implies also that

$$e_j(T^*T)T^*T e_j(T^*T) \subseteq (T e_j(T^*T))^* (T e_j(T^*T))$$

By result 4.5, we know that $T^*T e_j^2(T^*T)$ and $e_j(T^*T) T^*T e_j(T^*T)$ are equal on the intersection of their domains. But $e_j(T^*T)$ and $e_j^2(T^*T)$ are right multipliers of T^*T . So

$$T^*T e_j^2(T^*T) = e_j(T^*T) T^*T e_j(T^*T) \in \mathcal{L}(E)$$

Hence,

$$T^*T e_j^2(T^*T) = (T e_j(T^*T))^* (T e_j(T^*T))$$

Consequently,

$$\begin{aligned} \text{Ran}(e_j(T^*T) T^*) &\subseteq \overline{\text{Ran}(T e_j(T^*T))^*} = \overline{\text{Ran}(T e_j(T^*T))^* (T e_j(T^*T))} \\ &= \overline{\text{Ran}(T^*T e_j^2(T^*T))} \subseteq \overline{\text{Ran } T^*T} \end{aligned}$$

Choose $v \in D(T^*)$. It is clear that the net $(e_i(T^*T) T^*(v))_{i \in I}$ converges to $T^*(v)$. By the equation above, we know that $e_i(T^*T) T^*(v) \in \overline{\text{Ran } T^*T}$ for every $i \in I$. Hence, $T^*(v)$ belongs to $\overline{\text{Ran } T^*T}$. So we see that $\overline{\text{Ran } T^*} \subseteq \overline{\text{Ran } T^*T}$. The other inclusion is trivially true.

The equality involving the kernel is straightforward to prove. ■

5 Invertible regular operators

In the first part of this section, we introduce the proper notion of invertibility of regular operators and investigate its basic properties (see page 104 of [7]). In the second part, we look at the proper notion of bounded invertibility.

Definition 5.1 Consider Hilbert C^* -modules E, F over a C^* -algebra A and let T be an element in $\mathcal{R}(E, F)$. Then we call T invertible (in $\mathcal{R}(E, F)$) $\Leftrightarrow T$ is injective and T^{-1} belongs to $\mathcal{R}(F, E)$.

If T is invertible, then T^{-1} is densely defined so T has dense range. It is also immediately clear that T^{-1} is invertible if T is invertible.

As usual, we have the following results.

Lemma 5.2 Consider Hilbert C^* -modules E, F over a C^* -algebra A and let T be an element in $\mathcal{R}(E, F)$. Then we have the following properties :

- If T has dense range, then T^* is injective.
- If T has dense range and is injective, then $(T^{-1})^* = (T^*)^{-1}$.

Proof :

- Suppose that T has dense range. Take $v \in D(T^*)$ such that $T^*v = 0$. Then we have for every $w \in D(T)$ that $\langle v, Tw \rangle = \langle T^*v, w \rangle = 0$. Because T has dense range, this implies that $v = 0$.
- Suppose that T has dense range and that T is injective. We know already that T^* is injective.

1. Choose $v \in D(T^*)$. Then we have for every $w \in D(T)$ that

$$\langle T^{-1}(Tw), T^*v \rangle = \langle w, T^*v \rangle = \langle Tw, v \rangle .$$

So T^*v belongs to $D((T^{-1})^*)$ and $(T^{-1})^*(T^*v) = v$

This implies that $(T^*)^{-1} \subseteq (T^{-1})^*$.

2. Choose $v \in D((T^{-1})^*)$. Then we have for every $w \in D(T)$ that

$$\langle (T^{-1})^*(v), Tw \rangle = \langle v, T^{-1}(Tw) \rangle = \langle v, w \rangle .$$

So $(T^{-1})^*(v)$ belongs to $D(T^*)$ and $T^*((T^{-1})^*(v)) = v$.

This implies that $(T^{-1})^* \subseteq (T^*)^{-1}$.

So we get that $(T^{-1})^* = (T^*)^{-1}$. ■

Result 5.3 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and let T be an element in $\mathcal{R}(E, F)$. Then we have the following properties.*

- T is invertible $\Leftrightarrow T^*$ is invertible
- If T is invertible, then $(T^*)^{-1} = (T^{-1})^*$.

Proof : Suppose that T is invertible.

Because T has dense range and is injective, the previous lemma implies that T^* is injective and $(T^*)^{-1} = (T^{-1})^*$. Hence, the regularity of T^{-1} implies that $(T^*)^{-1}$ is regular.

The other implication in the first statement follows by symmetry. ■

Another useful characterization of invertibility is given in the following proposition.

Proposition 5.4 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and let T be an element in $\mathcal{R}(E, F)$. Then T is invertible $\Leftrightarrow T$ and T^* have dense range.*

Proof :

\Rightarrow This follows immediately from the previous proposition and the remark after definition 5.1.

\Leftarrow Because T^* has dense range, T is injective by lemma 5.2.

Because T has dense range, we have that T^{-1} is a densely defined closed A -linear operator from within F into E .

By lemma 5.2, we have moreover that $(T^{-1})^* = (T^*)^{-1}$. Because T^* has dense range, this implies that $(T^{-1})^*$ is densely defined.

Denote the flip map from $E \oplus F$ to $F \oplus E$ by U . Then theorem 9.3 of [7] implies that

$$G(T^{-1}) + G(T^{-1})^\perp = U(G(T)) + U(G(T))^\perp = U(G(T) + G(T)^\perp) = U(E \oplus F) = F \oplus E .$$

Hence, proposition 9.5 of [7] implies that T^{-1} is regular. ■

Because the image of z_T and T are the same, this proposition implies immediately the following result.

Corollary 5.5 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and let T be an element in $\mathcal{R}(E, F)$. Then T is invertible $\Leftrightarrow z_T$ is invertible (in $\mathcal{R}(E, F)!$).*

Another immediate corollary is the following one.

Corollary 5.6 *Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Then T is invertible $\Leftrightarrow T$ has dense range.*

Proof : Suppose that T has dense range. Then proposition 4.18 implies that TT^* has dense range. So T^*T has also dense range. Consequently has T^* also dense range. ■

This implies of course also that strictly positive elements are just the invertible positive elements.

Using proposition 4.18, we get immediately the next result.

Proposition 5.7 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and let T be an elements in $\mathcal{R}(E, F)$. Then T is invertible $\Leftrightarrow T^*T$ and TT^* are invertible.*

Invertibility is transferred by non-degenerate $*$ -homomorphisms :

Proposition 5.8 *Consider a Hilbert C^* -module E over a C^* -algebra A . Let B be a C^* -algebra and π a non-degenerate $*$ -homomorphism from B into $\mathcal{L}(E)$. Consider an invertible element T affiliated with B . Then $\pi(T)$ is invertible and $\pi(T)^{-1} = \pi(T^{-1})$.*

Proof : We have for every $b \in D(T)$ and $v \in E$ that $\pi(b)v$ belongs to $D(\pi(T))$ and $\pi(T)(\pi(b)v) = \pi(T(b))v$. Hence, the density of the range of T and the non-degeneracy of π imply that $\pi(T)$ has dense range. We get in a similar way that $\pi(T)^* = \pi(T^*)$ has dense range.

So we see that $\pi(T)$ is invertible.

Choose $b \in D(T^{-1})$ and $v \in E$. Then $\pi(b)v$ belongs to $D(\pi(T^{-1}))$ and $\pi(T^{-1})(\pi(b)v) = \pi(T^{-1}(b))v$.

Because $T^{-1}(b)$ belongs to $D(T)$, we get that $\pi(T^{-1}(b))v$ belongs to $D(\pi(T))$ and that

$$\pi(T)(\pi(T^{-1}(b))v) = \pi(T(T^{-1}(b)))v = \pi(b)v.$$

This implies that $\pi(b)v$ belongs to $D(\pi(T)^{-1})$ and $\pi(T)^{-1}(\pi(b)v) = \pi(T^{-1}(b))v = \pi(T^{-1})(\pi(b)v)$.

Consequently, the closedness of $\pi(T)^{-1}$ and the fact that the set $\langle \pi(b)v \mid b \in D(T^{-1}), v \in E \rangle$ is a core for $\pi(T^{-1})$ imply that $\pi(T^{-1}) \subseteq \pi(T)^{-1}$.

We have in a similar way that $\pi((T^*)^{-1}) \subseteq \pi(T^*)^{-1}$, which gives us that $(\pi(T)^{-1})^* \subseteq (\pi(T^*)^{-1})^*$. Hence $\pi(T)^{-1} \subseteq \pi(T^{-1})$ by taking the adjoint of the last inclusion.

So we arrive at the conclusion that $\pi(T)^{-1} = \pi(T^{-1})$. ■

Even if π is injective, the converse of this implication is not true : Define the non-degenerate $*$ -homomorphism π from $C([0, 1])$ into $C_0(]0, 1[)$ such that $\pi(f) = f_{]0, 1[}$ for every $f \in C([0, 1])$. Then π is injective but you can find easily a function $g \in C([0, 1])$ such that $\pi(g)$ is invertible but g isn't (cfr. the next result).

In the commutative case, this form of invertibility corresponds to the usual one.

Proposition 5.9 *Consider a locally compact space X and let f be an element in $C(X)$. Then we have the following properties :*

- We have that f is invertible $\Leftrightarrow f(x) \neq 0$ for every $x \in X$.
- If f is invertible, then $f^{-1}(x) = \frac{1}{f(x)}$ for every $x \in X$.

Proof :

- Suppose there exists $x_0 \in X$ such that $f(x_0) = 0$. Then it is clear that $g(x_0) = 0$ for every $g \in \overline{fC_0(X)}$. This implies that $fC_0(X)$ is not dense in $C_0(X)$. So f is not invertible.
- Suppose that $f(x) \neq 0$ for every $x \in X$.
Take $h \in K(X)$ and define $g \in K(X)$ such that $g(x) = \frac{h(x)}{f(x)}$ for every $x \in X$. Then g belongs to $D(f)$ and $f(g) = h$. This implies that f has dense range.
We get in a similar way that \bar{f} has dense range. So f is invertible.
Define the function $k \in C(X)$ such that $k(x) = \frac{1}{f(x)}$ for every $x \in X$. Then it is easy to check that $f k = \iota_{D(k)}$ and that $k f = \iota_{D(f)}$. So $f^{-1} = k$.

■

The following corollary will be useful in the next section.

Corollary 5.10 *Consider a locally compact space X and let f be an element in $C(X)$. Let c be a complex number. Then $c \notin f(X) \Leftrightarrow f - c1$ is invertible.*

We will now consider a stronger form of invertibility which is natural to look at in connection with the spectrum of an element.

Definition 5.11 *Consider two Hilbert C^* -modules E, F over a C^* -algebra A and let T be an element in $\mathcal{R}(E, F)$. Then we call T adjointable invertible \Leftrightarrow There exists an element $S \in \mathcal{L}(F, E)$ such that $ST \subseteq TS = 1$*

Corollary 5.12 *Consider two Hilbert C^* -modules E and F over a C^* -algebra A and let T be an element in $\mathcal{R}(E, F)$. Then T is adjointable invertible $\Leftrightarrow T$ is invertible and T^{-1} belongs to $\mathcal{L}(F, E)$.*

Remark 5.13 Let E be a Hilbert C^* -module over a C^* -algebra A and let T an element in $\mathcal{L}(E)$. Then it is clear that T is adjointable invertible if and only if T is invertible in the C^* -algebra $\mathcal{L}(E)$.

A useful characterization for adjointable invertibility is the following generalization of a well known Hilbert result.

Result 5.14 *Consider two Hilbert C^* -modules E and F over a C^* -algebra A and let T be an element in $\mathcal{R}(E, F)$. Then T is adjointable invertible $\Leftrightarrow \text{Ran } T = F$ and $\text{Ran } T^*$ is dense in $E \Leftrightarrow \text{Ran } T = F$ and $\text{Ran } T^* = E$*

Proof :

- Suppose that T is adjointable invertible. It is clear that $\text{Ran } T = F$.
Now T is invertible and T^{-1} belongs to $\mathcal{L}(F, E)$. We know that T^* is invertible and $(T^*)^{-1} = (T^{-1})^* \in \mathcal{L}(E, F)$, so $\text{Ran } T^* = F$.
- Suppose that $\text{Ran } T = F$ and $\text{Ran } T^*$ is dense in E . By 5.4, we know that T is invertible. Because $\text{Ran } T = F$, we get that $D(T^{-1}) = F$. Combining this with the closedness of T^{-1} , the closed graph theorem implies that T^{-1} is bounded.

We know also that T^* is invertible and that $(T^{-1})^* = (T^*)^{-1}$. Because $\text{Ran } T^*$ is dense in E , this implies that $(T^{-1})^*$ is densely defined.

Because T^{-1} is bounded, we have also that $(T^{-1})^*$ is bounded. So combining this with the closedness of $(T^{-1})^*$, we must have that $D((T^{-1})^*)$ closed. Hence, $D((T^{-1})^*) = E$.

From this all, we conclude that T^{-1} belongs to $\mathcal{L}(F, E)$.

■

This implies immediately the following result.

Corollary 5.15 *Consider two Hilbert C^* -modules E and F over a C^* -algebra A and let T be an element in $\mathcal{R}(E, F)$. Then T is adjointable invertible $\Leftrightarrow T^*$ is adjointable invertible.*

Combining result 5.14 with corollary 5.6, we get also the following one.

Corollary 5.16 *Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Then T is adjointable invertible $\Leftrightarrow \text{Ran } T = E$.*

We can again connect the adjoint invertibility of T to the invertibility of its z -transform. The proof of this result follows immediately from the fact that $\text{Ran } T = \text{Ran } z_T$ and $\text{Ran } T^* = \text{Ran } (z_T)^*$ and result 5.14.

Result 5.17 *Consider two Hilbert C^* -modules E and F over a C^* -algebra A and let T be an element in $\mathcal{R}(E, F)$. Then T is adjointable invertible $\Leftrightarrow z_T$ is adjointable invertible.*

This characterization of adjoint invertibility makes the proof of the following result very easy. However, a little bit of care has to be taken for the second equivalence. You need the fact that invertibility for an element in $M(\pi(B))$ is equivalent to the invertibility of this element in $\mathcal{L}(E)$.

Result 5.18 *Consider a Hilbert C^* -module E over a C^* -algebra A . Let B be a C^* -algebra and π a non-degenerate $*$ -homomorphism from B into $\mathcal{L}(E)$. Let T be a regular operator in E . Then we have the following properties.*

- *If T is adjointable invertible, then $\pi(T)$ is adjointable invertible.*
- *Suppose moreover that π is injective.*
Then T is adjointable invertible $\Leftrightarrow \pi(T)$ is adjointable invertible.

Another application of result 5.17 is the proof of the fact that the definition of the spectrum of a normal regular operator given by Woronowicz coincides with the usual definition in the Hilbert space case. First we need an easy lemma.

Lemma 5.19 *Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Then we have for every $c \in \mathbb{C}$ that $\sigma(T - c1) = \sigma(T) - c$.*

Proof : Define $f \in C(\mathbb{C})$ such that $f(\lambda) = \lambda - c$ for $\lambda \in \mathbb{C}$. Then corollary 4.11 implies that $f(T) = T - c1$. Hence,

$$\sigma(T - c1) = \overline{f(\sigma(T))} = \overline{\sigma(T) - c} = \sigma(T) - c.$$

■

Proposition 5.20 *Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Let c be a complex number. Then $c \notin \sigma(T) \Leftrightarrow T - c1$ is adjointable invertible*

Proof : Using the previous lemma, result 3.7 and result 5.17, we get that

$$\begin{aligned} c \notin \sigma(T) &\Leftrightarrow 0 \notin \sigma(T - c1) \Leftrightarrow 0 \notin \sigma(z_{T-c1}) \Leftrightarrow z_{T-c1} \text{ is invertible in } \mathcal{L}(E) \\ &\Leftrightarrow T - c1 \text{ is adjointable invertible} \end{aligned}$$

■

This suggests the following natural and consistent definition of the spectrum of a general regular operator.

Definition 5.21 Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. We define the resolvent of T as the set $\rho(T) = \{ c \in \mathbb{C} \mid T - c1 \text{ is adjointable invertible} \}$ and we define the spectrum of T as the set $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

The usual Hilbert space techniques allow us to prove the next two results.

Result 5.22 Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Then $\sigma(T)$ is closed and the mapping $\rho(T) \rightarrow \mathcal{L}(E) : c \mapsto (T - c)^{-1}$ is continuous.

Proof : Choose $d \in \rho(T)$. Define $M = \|(T - d1)^{-1}\| \in \mathbb{R}^+$.

We have for every $c \in \mathbb{C}$ that $(T - d1)^{-1}E = D(T - d1) = D(T - c1)$ so result 2.3 implies that $(T - c1)(T - d1)^{-1}$ belongs to $\mathcal{L}(E)$. Define the function f from \mathbb{C} into $\mathcal{L}(E)$ such that $f(c) = (T - c1)(T - d1)^{-1}$ for every $c \in \mathbb{C}$. Then $f(d) = 1$.

We have for every $c_1, c_2 \in \mathbb{C}$ that

$$\begin{aligned} \|f(c_1) - f(c_2)\| &= \|(T - c_11)(T - d1)^{-1} - (T - c_21)(T - d1)^{-1}\| \\ &= \|(c_2 - c_1)(T - d1)^{-1}\| = |c_1 - c_2| M \end{aligned}$$

so the function f is continuous.

Choose $c \in \mathbb{C}$ such that $|c - d| \leq \frac{1}{M+1}$. Then the previous inequality implies that $\|f(c) - 1\| < 1$ which in turn implies that $f(c)$ is invertible in $\mathcal{L}(E)$.

It is clear that $f(c)(T - d1) = T - c1$. Hence we get that $T - c1$ is injective and that

$$(T - c1)^{-1} = (T - d1)^{-1} f(c)^{-1}$$

So $(T - c1)^{-1}$ belongs to $\mathcal{L}(E)$. By definition, we have that c belongs to $\rho(T)$.

The above equation and the continuity of the invertibility operation in $\mathcal{L}(E)$ imply also immediately that the function $\rho(T) \rightarrow \mathcal{L}(E) : c \mapsto (T - c)^{-1}$ is continuous. ■

Result 5.23 Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Then the mapping $\rho(T) \rightarrow \mathcal{L}(E) : c \mapsto (T - c)^{-1}$ is analytic.

As usual, this follows from the fact that $(T - c1)^{-1} - (T - d1)^{-1} = (c - d)(T - c1)^{-1}(T - d1)^{-1}$ for $c, d \in \rho(T)$.

Result 5.18 implies also the following result.

Result 5.24 Consider a Hilbert C^* -module E over a C^* -algebra A . Let B be a C^* -algebra and π a non-degenerate $*$ -homomorphism from B into $\mathcal{L}(E)$. Let T be a regular operator in E . Then we have the following properties.

- $\sigma(\pi(T)) \subseteq \sigma(T)$
- If π is injective, then $\sigma(\pi(T)) = \sigma(T)$.

6 The functional calculus revisited

Consider a normal regular operator T in a Hilbert C^* -module. Until now, we had a functional calculus for T on the spectrum $\sigma(T)$. But there are cases where we want a functional calculus for T on $\sigma(T) \setminus K$ where K is a certain well-behaved finite subset of \mathbb{C} .

A typical situation occurs when T is strictly positive in which case we want to define any complex power of T . So we need in this case a functional calculus on $\sigma(T) \setminus \{0\}$.

We will always work with functional calculi on a special kind of subsets of \mathbb{C} . Therefore we will use the following terminology.

Terminology 6.1 *Let G be a subset of \mathbb{C} . Then we call G almost closed if there exists a finite subset K of \mathbb{C} such that $G \cup K$ is closed.*

It is clear that any almost closed subset is locally compact. By adding or subtracting a finite number of points of an almost closed set, we keep an almost closed set.

Lemma 6.2 *Let G be an almost closed subset of \mathbb{C} and $f \in C_0(G)$. Then there exists an element $g \in C_0(\mathbb{C})$ such that $f \subseteq g$.*

Proof : There exists a finite subset K of $\mathbb{C} \setminus G$ such that $G \cup K$ is closed. Define $C = \{h_{G \cup K} \mid h \in C_0(\mathbb{C})\}$. Because $G \cup K$ is closed, C will be a subset of $C_0(G \cup K)$. So C is a sub C^* -algebra $C_0(G \cup K)$ (as the image under an obvious $*$ -homomorphism). We have also clearly that

- There exists for every $c \in G \cup K$ an element $k \in C$ such that $k(c) \neq 0$.
- There exists for every $c, d \in G \cup K$ with $c \neq d$, an element $k \in C$ such that $k(c) \neq k(d)$.

Hence, the Stone Weierstrass theorem implies that $C = C_0(G \cup K)$.

Now define $\tilde{f} \in C_0(G \cup K)$ such that $\tilde{f} = 0$ on F and $\tilde{f} = f$ on G (the closedness of F is here used to get the continuity of \tilde{f} in points of G). Then there exists $g \in C_0(\mathbb{C})$ such that $g_{G \cup K} = \tilde{f}$. Hence, $f \subseteq g$. ■

Result 6.3 *Consider a Hilbert C^* -module over a C^* -algebra A and an almost closed subset G of \mathbb{C} and $f \in C_0(G)$. Let π, θ be non-degenerate $*$ -homomorphisms from $C_0(G)$ into $\mathcal{L}(E)$. If $\pi(\iota_G) = \theta(\iota_G)$, then $\pi = \theta$.*

Proof : By proposition 3.9, we have for every $g \in C_0(\mathbb{C})$ that $\pi(g_G) = \pi(g \circ \iota_G) = \pi(g(\iota_G)) = g(\pi(\iota_G))$ and similarly, $\theta(g_G) = g(\pi(\iota_G))$. So we see that $\pi(g_G) = \theta(g_G)$. Now the previous lemma implies that $\pi = \theta$. ■

We will soon prove the key result to introduce these new functional calculi but first we need the following lemma.

Lemma 6.4 *Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Consider $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ such that $T - \lambda_1 1, \dots, T - \lambda_n 1$ have dense range. Define the function g from $C(\sigma(T))$ such that $g(c) = (c - \lambda_1) \dots (c - \lambda_n)$ for every $c \in \sigma(T)$. Then $g(T)$ has dense range.*

Proof : Take an approximate unit $(e_k)_{k \in K}$ for $C_0(\sigma(T))$ in $K(\sigma(T))$.

Define for every $r \in \{1, \dots, n\}$ the function $g_r \in C(\sigma(T))$ such that $g_r(c) = (c - \lambda_1) \dots (c - \lambda_r)$ for every $c \in \sigma(T)$.

1. Because $g_1(T) = T - \lambda_1 1$, we have by assumption that $g_1(T)$ has dense range.
2. Choose $r \in \{1, \dots, n-1\}$ and suppose that $g_r(T)$ has dense range.

Define the function $h \in C(\sigma(T))$ such that $h(c) = c - \lambda_{r+1}$ for every $c \in \sigma(T)$. Then $h(T) = T - \lambda_{r+1} 1$, so $h(T)$ has dense range.

Take $k \in K$. By lemma 4.3, we know that $e_k(T)E \subseteq D(g_{r+1}(T))$. We have moreover that

$$\begin{aligned} \overline{g_{r+1}(T)(e_k(T)E)} &= \overline{(g_{r+1}e_k)(T)E} = \overline{(g_{r+1}e_k)(T)D(h(T))} \\ &= \overline{(g_r h e_k)(T)D(h(T))} = \overline{(g_r e_k)(T)(h(T)D(h(T)))}. \end{aligned}$$

So, using the fact that $h(T)$ has dense range, we see that

$$\begin{aligned} \overline{g_{r+1}(T)(e_k(T)E)} &= \overline{(g_r e_k)(T)E} \supseteq (g_r e_k)(T)E \\ &\supseteq (g_r e_k)(T)D(g_r(T)) = e_k(T)(g_r(T)D(g_r(T))). \end{aligned}$$

Hence, the assumed density of the range of $g_r(T)$ implies that

$$e_k(T)E \subseteq \overline{g_{r+1}(T)(e_k(T)E)} \subseteq \overline{\text{Ran } g_{r+1}(T)}.$$

Because $(e_k(T))_{k \in K}$ converges strongly to 1, this implies that $\overline{\text{Ran } g_{r+1}(T)} = E$.

By induction, we can now conclude that $g(T) = g_n(T)$ has dense range. ■

Now we are ready to prove the main result of this section.

Proposition 6.5 *Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Consider a finite subset K of \mathbb{C} such that $T - \lambda 1$ is invertible for every $\lambda \in K$. Then there exists a unique non-degenerate $*$ -homomorphism π from $C_0(\sigma(T) \setminus K)$ into $\mathcal{L}(E)$ such that $\pi(\iota_{\sigma(T) \setminus K}) = T$. We have moreover for every $f \in C(\sigma(T))$ that $f(T) = \pi(f_{\sigma(T) \setminus K})$.*

Proof : The unicity is guaranteed by result 6.3 so let us turn to the existence.

There is nothing to prove if $G \cap K = \emptyset$ so suppose that $G \cap K \neq \emptyset$. Then there exist different complex numbers $\lambda_1, \dots, \lambda_n$ such that $G \cap K = \{\lambda_1, \dots, \lambda_n\}$. Then $G \setminus K = G \setminus \{\lambda_1, \dots, \lambda_n\}$ and we have for every $i \in \{1, \dots, n\}$ that $T - \lambda_i 1$ has dense range.

For every $f \in C_0(\sigma(T) \setminus \{\lambda_1, \dots, \lambda_n\})$, we define the function $\tilde{f} \in C_0(\sigma(T))$ such that

$$\tilde{f} = f \text{ on } \sigma(T) \setminus \{\lambda_1, \dots, \lambda_n\} \quad \text{and} \quad \tilde{f}(\lambda_i) = 0 \text{ for every } i \in \{1, \dots, n\}.$$

Then the mapping $C_0(\sigma(T) \setminus \{\lambda_1, \dots, \lambda_n\}) \rightarrow C_0(\sigma(T)) : f \mapsto \tilde{f}$ is clearly a $*$ -homomorphism.

Now define the mapping π from $C_0(\sigma(T) \setminus \{\lambda_1, \dots, \lambda_n\})$ into $\mathcal{L}(E)$ such that $\pi(f) = \tilde{f}(T)$ for every $f \in C_0(\sigma(T) \setminus \{\lambda_1, \dots, \lambda_n\})$. Then π is a $*$ -homomorphism.

- First we prove that π is non-degenerate.

Define the function $g \in C(\sigma(T))$ such that $g(c) = (c - \lambda_1) \dots (c - \lambda_n)$ for every $c \in \sigma(T)$. By the previous lemma, we know that $g(T)$ has dense range.

Take an approximate unit $(e_k)_{k \in K}$ for $C_0(\sigma(T))$ in $K(\sigma(T))$.

Take $k \in K$ and define h_k as the restriction of $g e_k$ to $\sigma(T) \setminus \{\lambda_1, \dots, \lambda_n\}$.

Because $g(\lambda_1) = \dots = g(\lambda_n) = 0$, we have that h_k belongs to $C_0(\sigma(T) \setminus \{\lambda_1, \dots, \lambda_n\})$ and that $\widetilde{h_k} = g e_k$, so $\pi(h_k) = (g e_k)(T)$.

This implies that

$$e_k(T)(g(T)D(g(T))) = (g e_k)(T)D(g(T)) = \pi(h_k)D(g(T)) \subseteq \pi(C_0(\sigma(T) \setminus \{\lambda_1, \dots, \lambda_n\})) E$$

So the density of the range of $g(T)$ implies that $e_k(T)E \subseteq \overline{\pi(C_0(\sigma(T) \setminus \{\lambda_1, \dots, \lambda_n\})) E}$.

Because $(e_k(T))_{k \in K}$ converges strongly to 1, this implies that $\pi(C_0(\sigma(T) \setminus \{\lambda_1, \dots, \lambda_n\})) E$ is dense in E .

- Define the mapping θ from $C_0(\sigma(T))$ into $C_b(\sigma(T) \setminus \{\lambda_1, \dots, \lambda_n\})$ such that we have for every $h \in C_0(\sigma(T))$ that $\theta(h)$ is the restriction of h to $\sigma(T) \setminus \{\lambda_1, \dots, \lambda_n\}$.

Then θ is non-degenerate and we have for every $h \in C(\sigma(T))$ that $\theta(h)$ is the restriction of h to $\sigma(T) \setminus \{\lambda_1, \dots, \lambda_n\}$.

Choose $g \in C_0(\sigma(T))$.

Take an approximate unit $(e_k)_{k \in K}$ for $C_0(\sigma(T) \setminus \{\lambda_1, \dots, \lambda_n\})$ in $K(\sigma(T) \setminus \{\lambda_1, \dots, \lambda_n\})$.

Choose $k \in K$. Then $\theta(g) e_k$ belongs to $K(\sigma(T) \setminus \{\lambda_1, \dots, \lambda_n\})$ and $(\theta(g) e_k)^\sim = g \widetilde{e_k}$.

So we get that

$$\pi(\theta(g)) \pi(e_k) = \pi(\theta(g) e_k) = (g \widetilde{e_k})(T) = g(T) \widetilde{e_k}(T) = g(T) \pi(e_k).$$

Because $(\pi(e_k))_{k \in K}$ converges strongly to 1, we get that $\pi(\theta(g)) = g(T) = \psi_T(g)$.

Hence, proposition 1.11 implies for every $f \in C(\sigma(T))$ that $\pi(\theta(f)) = (\pi\theta)(f) = \psi_T(f) = f(T)$.

We find in particular that $\pi(\iota_{\sigma(T) \setminus \{\lambda_1, \dots, \lambda_n\}}) = \pi(\theta(\iota_{\sigma(T)})) = \iota_{\sigma(T)}(T) = T$.

■

The rest of this section is devoted to introduce the proper notations and prove quickly the usual results for functional calculi. The proofs of these results do not differ very much of the proofs of their corresponding results for the ordinary functional calculus.

Terminology 6.6 Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Let f be a function from within \mathbb{C} into \mathbb{C} . Then we call f compatible with $T \Leftrightarrow f$ is continuous and there exists a finite subset K of \mathbb{C} such that $T - \lambda 1$ is invertible for every $\lambda \in K$ and such that f is defined on $\sigma(T) \setminus K$.

We want to define $f(T)$ for such compatible functions f . First we have to resolve a small consistency problem. This will be done in the following lemma.

Lemma 6.7 Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Consider finite subsets K, L of \mathbb{C} such that $T - \lambda 1$ is invertible for every $\lambda \in K \cup L$.

- Let π denote the unique non-degenerate $*$ -homomorphism π from $C_0(\sigma(T) \setminus K)$ into $\mathcal{L}(E)$ such that $\pi(\iota_{\sigma(T) \setminus K}) = T$.

- Let θ denote the unique non-degenerate $*$ -homomorphism π from $C_0(\sigma(T) \setminus L)$ into $\mathcal{L}(E)$ such that $\theta(\iota_{\sigma(T) \setminus L}) = T$.

Then we have for every continuous function f from within \mathbb{C} into \mathbb{C} which is defined on both $\sigma(T) \setminus K$ and $\sigma(T) \setminus L$ that $\pi(f_{\sigma(T) \setminus K}) = \theta(f_{\sigma(T) \setminus L})$.

Proof : Let η denote the unique non-degenerate $*$ -homomorphism π from $C_0(\sigma(T) \setminus (K \cup L))$ into $\mathcal{L}(E)$ such that $\eta(\iota_{\sigma(T) \setminus (K \cup L)}) = T$.

Define ρ to be the $*$ -homomorphism from $C_0(\sigma(T) \setminus K)$ into $C_b(\sigma(T) \setminus (K \cup L))$ such that we have for every $g \in C_0(\sigma(T) \setminus K)$ that $\rho(g)$ is the restriction of g to $\sigma(T) \setminus (K \cup L)$. Then ρ is non-degenerate and we have for every $g \in C(\sigma(T) \setminus K)$ that $\rho(g)$ is the restriction of g to $\sigma(T) \setminus (K \cup L)$.

So we have that $(\eta\rho)(\iota_{\sigma(T) \setminus K}) = \eta(\rho(\iota_{\sigma(T) \setminus K})) = \eta(\iota_{\sigma(T) \setminus (K \cup L)}) = T$, which implies that $\pi = \eta\rho$

This implies that $\pi(f_{\sigma(T) \setminus K}) = \eta(\rho(f_{\sigma(T) \setminus K})) = \eta(f_{\sigma(T) \setminus (K \cup L)})$.

We get in a similar way that $\theta(f_{\sigma(T) \setminus L}) = \eta(f_{\sigma(T) \setminus (K \cup L)})$. So the lemma follows. \blacksquare

Proposition 6.5 (or the previous lemma with $L = \emptyset$) implies moreover the following lemma.

Lemma 6.8 Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Consider a finite subset K of \mathbb{C} such that $T - \lambda 1$ is invertible for every $\lambda \in K$.

Let π denote the unique non-degenerate $*$ -homomorphism π from $C_0(\sigma(T) \setminus K)$ into $\mathcal{L}(E)$ such that $\pi(\iota_{\sigma(T) \setminus K}) = T$. Then we have for every continuous function f from within \mathbb{C} into \mathbb{C} which is defined on $\sigma(T)$ that $\pi(f_{\sigma(T) \setminus K}) = f(T)$.

Now we are in a position to give the following consistent definition.

Definition 6.9 Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Let f be a function from within \mathbb{C} into \mathbb{C} which is compatible with T . Then we define the normal element $f(T)$ in $\mathcal{R}(E)$ as follows.

Take a finite subset K of \mathbb{C} such that $T - \lambda 1$ is invertible for every $\lambda \in K$ and such that f is defined on $\sigma(T) \setminus K$. Let π denote the unique non-degenerate $*$ -homomorphism π from $C_0(\sigma(T) \setminus K)$ into $\mathcal{L}(E)$ such that $\pi(\iota_{\sigma(T) \setminus K}) = T$.

Then we define $f(T) = \pi(f_{\sigma(T) \setminus K})$.

If f is a bounded function, it is clear that $f(T)$ belongs to $\mathcal{L}(E)$.

Terminology 6.10 Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Let G be a subset of \mathbb{C} . Then we say that G is compatible with T if and only if G is almost closed and there exist a finite subset K of \mathbb{C} such that

1. The element $T - \lambda 1$ is invertible for every $\lambda \in K$.
2. $\sigma(T) \setminus K \subseteq G$

Remark 6.11 The following properties are immediately clear.

- The sets $\sigma(T)$ and \mathbb{C} are always compatible with T .
- An almost closed subset of \mathbb{C} which contains a subset of \mathbb{C} which is compatible with T is itself compatible with T .

- If T is invertible, then $\mathbb{C} \setminus \{0\}$ and $\sigma(T) \setminus \{0\}$ are compatible with T .
- If T is strictly positive, then \mathbb{R}_0^+ is compatible with T .

If G is compatible with T , it is also clear that any element $f \in C(G)$ is compatible with T and so we can look at the normal element $f(T)$ in $\mathcal{R}(E)$.

Proposition 6.12 *Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Let G be a subset of \mathbb{C} which is compatible with T .*

Define the mapping π from $C_0(G)$ into $\mathcal{L}(E)$ such that $\pi(f) = f(T)$ for every $f \in C_0(G)$.

Then π is the unique non-degenerate $$ -homomorphism from $C_0(G)$ into $\mathcal{L}(E)$ such that $\pi(\iota_G) = T$. We call π the functional calculus of T on G . We have moreover that $\pi(f) = f(T)$ for every $f \in C(G)$.*

Proof : Take a finite subset K of \mathbb{C} such that

1. The element $T - \lambda 1$ is invertible for every $\lambda \in K$.
2. $\sigma(T) \setminus K \subseteq G$

Let θ denote the unique non-degenerate $*$ -homomorphism from $C_0(\sigma(T) \setminus K)$ into $\mathcal{L}(E)$ such that $\theta(\iota_{\sigma(T) \setminus K}) = T$.

Define the non-degenerate $*$ -homomorphism η from $C_0(G)$ into $C_b(\sigma(T) \setminus K)$ such that we have for every $f \in C_0(G)$ that $\eta(f)$ is equal to the restriction of f to $\sigma(T) \setminus K$. Then we have for every $f \in C(G)$ that $\eta(f)$ is equal to the restriction of f to $\sigma(T) \setminus K$.

Then we have by definition for every $f \in C_0(G)$ that $(\theta\eta)(f) = \theta(\eta(f)) = f(T) = \pi(f)$ so $\theta\eta = \pi$.

This implies for every $f \in C(G)$ that $\pi(f) = \theta(\eta(f)) = f(T)$. We have in particular that $\pi(\iota_G) = \theta(\eta(\iota_G)) = \theta(\iota_{\sigma(T) \setminus K}) = T$. ■

Looking at definition 6.9 and terminology 6.10, we get immediately the following result.

Result 6.13 *Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Let G, H subsets of \mathbb{C} which are compatible with T and such that $G \subseteq H$. Then we have for every $f \in C(H)$ that $f(T) = (f_G)(T)$.*

We have of course the following injectivity property.

Result 6.14 *Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Let G be a subset of \mathbb{C} which is compatible with T and such that $G \subseteq \sigma(T)$. Then the functional calculus of T on G is injective.*

Proof : Call π the functional calculus of T on G . Take $f \in C_0(G)$ such that $\pi(f) = 0$. Define $g \in C_0(\sigma(T))$ such that $g = f$ on G and $g = 0$ on $\sigma(T) \setminus G$. Then $\psi_T(g) = \pi(f) = 0$, so the injectivity of ψ_T implies that $g = 0$, so $f = 0$. ■

Every compatible subset of \mathbb{C} gives rise to a compatible subset of \mathbb{C} of the kind above.

Result 6.15 *Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Let G be a subset of \mathbb{C} which is compatible with T . Then $G \cap \sigma(T)$ is compatible with T .*

Proof : Because G is compatible with T , there exists a finite subset K of \mathbb{C} such that

1. The element $T - \lambda 1$ is invertible for every $\lambda \in K$.
2. $\sigma(T) \setminus K \subseteq G$

Then it is clear that $G \cap \sigma(T) = \sigma(T) \setminus K$, so $G \cap \sigma(T)$ is compatible with T . ■

Result 6.16 *Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Let G be a subset of \mathbb{C} which is compatible with T and f an element in $C(G)$. Then $\sigma(f(T)) = \overline{f(G \cap \sigma(T))} \subseteq \overline{f(G)}$.*

Proof : Call π the functional calculus of T on $G \cap \sigma(T)$. Then result 6.14 implies that π is faithful. Put $g = f_{G \cap \sigma(T)} \in \mathcal{C}(G \cap \sigma(T))$. Then $f(T) = g(T) = \pi(g)$. By the remark after proposition 3.9, this implies that $\sigma(f(T)) = \sigma(g(T)) = \sigma(\pi(g)) = \overline{g(G \cap \sigma(T))} = \overline{f(G \cap \sigma(T))}$. ■

Proposition 6.17 *Consider a Hilbert C^* -module over a C^* -algebra A . Let B be a C^* -algebra and π a non-degenerate $*$ -homomorphism from B into $\mathcal{L}(E)$. Consider a normal element T affiliated with B and G a subset of \mathbb{C} which is compatible with T . Then we have the following properties.*

- G is compatible with $\pi(T)$
- We have for every $f \in C(G)$ that $f(\pi(T)) = \pi(f(T))$

Proof : Take a finite subset K of \mathbb{C} such that

1. The element $T - \lambda 1$ is invertible for every $\lambda \in K$.
2. $\sigma(T) \setminus K \subseteq G$

By proposition 5.8, we have for every $\lambda \in K$ that $\pi(T) - \lambda 1$ is invertible. Furthermore, proposition 3.9 implies that $\sigma(\pi(T)) \subseteq \sigma(T)$ so $\sigma(\pi(T)) \setminus K \subseteq \sigma(T) \setminus K \subseteq G$. So G is compatible with $\pi(T)$.

Let θ denote the functional calculus of T on G . Then we have that $(\pi\theta)(\iota_G) = \pi(\theta(\iota_G)) = \pi(T)$. So $\pi\theta$ is the functional calculus of $\pi(T)$ on G .

This implies that

$$f(\pi(T)) = (\pi\theta)(f) = \pi(\theta(f)) = \pi(f(T))$$

for every $f \in C(G)$. ■

Result 6.18 *Consider a locally compact space X and f an element in $C(X)$. Let G be an almost closed subset of \mathbb{C} . Then the following holds.*

- G is compatible with $f \Leftrightarrow f(X) \subseteq G$
- Suppose that G is compatible with f . Then we have that $g(f) = g \circ f$ for every $g \in C(G)$.

Proof : Suppose that G is compatible with f . Then there exists a finite K of \mathbb{C} such that

- We have for every $\lambda \in \mathbb{C}$ that $f - \lambda 1$ is invertible.
- $\sigma(f) \setminus K \subseteq G$.

By corollary 5.10, we know for every $\lambda \in K$ that $\lambda \notin f(X)$. Because moreover $\sigma(f) = \overline{f(X)}$, we get that $f(X) \subseteq \sigma(f) \setminus K$. So $f(X) \subseteq G$.

Define the *-homomorphism π from $C_0(G)$ into $C_b(X)$ such that $\pi(g) = g \circ f$ for $g \in C_0(G)$. Then π is non-degenerate and $\pi(g) = g \circ f$ for $g \in C(G)$.

We have in particular that $\pi(\iota_G) = f$, so π is the functional calculus of f on G .

Suppose on the other hand that $f(X) \subseteq G$. Because G is almost closed, there exists a finite subset K of $\mathbb{C} \setminus G$ such that $G \cup K$ is closed. Hence, $\sigma(f) = \overline{f(X)} \subseteq G \cup K$. Therefore, $\sigma(f) \setminus K \subseteq G$.

We have for every $\lambda \in K$ that $\lambda \notin G$, so $\lambda \notin f(X)$ which by corollary 5.10 implies that $f - \lambda 1$ is invertible. So we see that G is compatible with f . ■

If we combine the previous result with proposition 6.17, we get the following familiar one.

Proposition 6.19 *Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Let F be a subset of \mathbb{C} which is compatible with T and f an element in $C(F)$. Consider an almost closed subset G of \mathbb{C} such that $f(F) \subseteq G$. Then*

- *The set G is compatible with $f(T)$.*
- *We have that $g(f(T)) = (g \circ f)(T)$ for every $g \in C(G)$.*

Remark 6.20 Consider a Hilbert space H and a normal element T in $\mathcal{R}(H)$. Let G be a subset of \mathbb{C} which is compatible with H . Then G is certainly Borel measurable. Let $\mathcal{B}(G)$ denote the collection of Borel subsets of G .

Denote the spectral measure of T on \mathbb{C} by F and call E the restriction of F to $\mathcal{B}(G)$.

By assumption, there exists a finite subset K of \mathbb{C} such that

- We have for every $\lambda \in K$ that $T - \lambda 1$ is invertible.
- $\sigma(T) \setminus K \subseteq G$

We have for every $\lambda \in K$ that λ is not an eigenvalue of T which implies that $F(\{\lambda\}) = 0$. Because K is finite, this implies that $F(K) = 0$. Hence we get that $F((G \setminus \sigma(T)) \cup (\sigma(T) \setminus G)) = 0$.

So E is a spectral measure on G such that $\int \iota_G dE = T$. We call E the spectral measure of T on G .

Proposition 6.21 *Consider a Hilbert space H and T a normal operator in H . Let G be a subset of \mathbb{C} which is compatible with T and call E the spectral measure of T on G . Then $f(T) = \int f dE$ for every $f \in C(G)$.*

Proof : Define the *-homomorphism π from $C_0(G)$ into $\mathcal{B}(H)$ such that $\pi(f) = \int f dE$ for $f \in C_0(G)$. Take an approximate unit $(e_i)_{i \in I}$ for $C_0(G)$ in $K(G)$.

- First we prove that π is non degenerate.

Choose $v \in H$. Then we have a regular Borel measure E_v on G .

The set $\{e_i \mid i \in I\}$ is an upwardly directed set of continuous positive functions such that the equality $\sup \{e_i(c) \mid i \in I\} = 1$ holds for every $c \in G$.

Hence, the regularity of E_v implies that $\sup \{\int e_i dE_v \mid i \in I\} = \int 1 dE_v = \|v\|^2$.

So we see that $(\int e_i dE_v)_{i \in I}$ converges to 1.

We have moreover for every $i \in I$ that

$$\begin{aligned}\|v - \pi(e_i)v\|^2 &= \|v - (\int e_i dE) v\|^2 = \|(\int 1 - e_i dE) v\|^2 \\ &= \int |1 - e_i|^2 dE_v \leq \int |1 - e_i| dE_v = \int 1 - e_i dE_v = 1 - \int e_i dE_v\end{aligned}$$

This implies that $(\pi(e_i)v)_{i \in I}$ converges to v .

- Take $g \in C(G)$. We will prove now that $\pi(g) = \int g dE$.

Fix $j \in I$. Then the spectral functional calculus rules imply that $(\int e_j dE) H \subseteq D(\int g dE)$ and that

$$(\int e_j dE) (\int g dE) \subseteq (\int g dE) (\int e_j dE) = \int g e_j dE$$

This implies immediately that $\pi(e_j) H \subseteq D(\int g dE)$ and that

$$\pi(e_j) (\int g dE) \subseteq (\int g dE) \pi(e_j) = \pi(g e_j)$$

By the functional calculus rules of section 4, we have moreover that $\pi(e_j) H \subseteq \pi(g) H$ and that

$$\pi(e_j) \pi(g) \subseteq \pi(g) \pi(e_j) = \pi(g e_j)$$

Choose $v \in D(\int g dE)$. By the remarks above, we know for every $i \in I$ that $\pi(e_i)v \in D(\pi(g))$ and that

$$\pi(g)(\pi(e_i)v) = \pi(g e_i)v = \pi(e_i) (\int g dE) v$$

This implies that $(\pi(e_i)v)_{i \in I}$ converges to v and that $(\pi(g)(\pi(e_i)v))_{i \in I}$ converges to $(\int g dE) v$.

So the closedness of $\pi(g)$ implies that $v \in D(\pi(g))$ and $\pi(g)v = (\int g dE) v$.

Consequently, $\pi(g) \subseteq \int g dE$. We get in a similar way that $\int g dE \subseteq \pi(g)$. Hence, $\int g dE = \pi(g)$.

So we have in particular that $\pi(\iota_G) = \int \iota_G dE = T$ which implies that π is the functional calculus of T on G . ■

7 Powers of positive regular operators

In this section, we show that the power theory of positive closed operators has an obvious generalization to a power theory of positive regular operators. By the previous sections, the proofs of the results in this section can be copied from the corresponding proofs in the Hilbert space case and are always simple consequences of calculation rules from sections 4 and 6. As a consequence, this section contains not many proofs

For the sake of completeness, we will give a list of the basic calculation rules.

For the most part of this section, we will fix a Hilbert C^* -module E over a C^* -algebra A .

Let us start first with a basic result.

Result 7.1 *Consider a normal regular operator T in E . Let $n \in \mathbb{N} \cup \{0\}$ and define the continuous function f from \mathbb{C} into \mathbb{C} such that $f(c) = c^n$ for every $c \in \mathbb{C}$. Then $T^n = f(T)$.*

Proof : We proceed by induction.

Define for every $m \in \mathbb{N} \cup \{0\}$ the function f_m from \mathbb{C} into \mathbb{C} such that $f_m(c) = c^m$ for every $c \in \mathbb{C}$.

- We have that $f_0(T) = 1 = T^0$.
- Let $m \in \mathbb{N} \cup \{0\}$ and suppose that $f_m(T) = T^m$.

By definition of the functional calculus, the equality $f_1(T) = T$ holds.

We have that $f_{m+1}(T) = (f_m f_1)(T) \supseteq f_m(T) f_1(T) = T^m T = T^{m+1}$. Using lemma 4.15, it is not so difficult to see that $D(f_{m+1}(T)) \subseteq D(f_1(T))$. So we get that

$$\begin{aligned} D(T^{m+1}) &= D(f_m(T) f_1(T)) = D((f_m f_1)(T)) \cap D(f_1(T)) \\ &= D(f_{m+1}(T)) \cap D(f_1(T)) = D(f_{m+1}(T)). \end{aligned}$$

Consequently, $f_{m+1}(T) = T^{m+1}$.

■

Corollary 7.2 *Consider a normal regular operator T in E and $n \in \mathbb{N} \cup \{0\}$. Then T^n is a normal regular operator in E .*

Result 7.3 *Consider an invertible normal regular operator T in E . Let $n \in \mathbb{Z}$ and define the continuous function f from \mathbb{C}_0 into \mathbb{C}_0 such that $f(c) = c^n$ for every $c \in \mathbb{C}_0$. Then $T^n = f(T)$.*

Proof : If $n \geq 0$, then the result follows from the previous result. Therefore suppose that $n < 0$. Define the continuous function g, h from \mathbb{C}_0 into \mathbb{C}_0 such that $g(c) = \frac{1}{c}$ and $h(c) = c^{-n}$ for every $c \in \mathbb{C}_0$. Then $h \circ g = f$, so $f(T) = h(g(T))$ by proposition 6.19.

Hence, the previous result implies that $f(T) = g(T)^{-n}$. (*)

Define the continuous function k on \mathbb{C}_0 such that $k(c) = c$ for every $c \in \mathbb{C}_0$, so $k(T) = T$. We have moreover that $gk = 1$. By results 4.5 and 4.7, this implies that

$$g(T) T = g(T) k(T) \subseteq (gk)(T) = 1$$

and

$$D(g(T) T) = D(g(T) k(T)) = D((gk)(T)) \cap D(k(T)) = E \cap D(T) = D(T)$$

We get in a similar way that $T g(T) \subseteq 1$ and $D(T g(T)) = D(g(T))$.

So we see that $g(T) = T^{-1}$. By (*), this gives us that $f(T) = (T^{-1})^{-n} = T^n$.

■

Corollary 7.4 *Consider an invertible normal regular operator T in E and $n \in \mathbb{Z}$. Then T^n is an invertible normal regular operator in E .*

Using these results and the results of the previous sections, we are now able to define powers of positive affiliated elements and prove the most important results.

Definition 7.5 *Consider a positive regular operator T in E . Let s be a positive number and define the continuous function f from \mathbb{R}^+ into \mathbb{R}^+ such that $f(t) = t^s$ for every $t \in \mathbb{R}^+$. We define $T^s = f(T)$, so T^s is a positive element affiliated with A .*

Due to result 7.1, this definition is consistent with the usual notion of T^s when s belongs to $\mathbb{N} \cup \{0\}$. Due to proposition 6.21, this definition is also consistent with the usual definition of T^s if E is a Hilbert space.

The following calculation rules hold:

Proposition 7.6 *Consider a positive regular operator T in E . Then the following elementary properties hold:*

1. *We have for every $s, t \in \mathbb{R}^+$ that $(T^s)^t = T^{st}$.*
2. *We have for every $s, t \in \mathbb{R}^+$ with $s \leq t$ that $D(T^t) \subseteq D(T^s)$.*
3. *We have for every $s, t \in \mathbb{R}^+$ that $T^{s+t} = T^s T^t$.*

The first result follows from proposition 6.19 and the second from lemma 4.15. The third equality follows from results 4.5 and 4.7, using the second result.

Remark 7.7 Let S, T be positive regular operators in E and r a strictly positive number. The first result of the previous proposition implies that $S = T \Leftrightarrow S^r = T^r$.

Therefore, we have for every regular positive operator T in E that $T^{\frac{1}{2}}$ is the unique positive element in $\mathcal{R}(E)$ such that $(T^{\frac{1}{2}})^2 = T$.

Result 7.8 *Consider a Hilbert C^* -modules E over a C^* -algebra A and a positive element $T \in \mathcal{R}(E)$. Then we have for every $\alpha \in \mathbb{R}_0^+$ that $\overline{\text{Ran } T} = \overline{\text{Ran } T^\alpha}$.*

Proof : By proposition 4.18, we have that $\overline{\text{Ran } T} = \overline{\text{Ran } T^{2^n}}$ for every $n \in \mathbb{Z}$.

We have for $\beta, \gamma \in \mathbb{R}_0^+$ with $\beta \leq \gamma$ that $T^\gamma = T^{\beta+(\gamma-\beta)} = T^\beta T^{\gamma-\beta}$ which implies that $\overline{\text{Ran } T^\gamma} \subseteq \overline{\text{Ran } T^\beta}$. Suppose that $\alpha \geq 1$. Then there exists $m \in \mathbb{N}$ such that $\alpha \leq 2^m$. So we get that

$$\overline{\text{Ran } T} = \overline{\text{Ran } T^{2^m}} \subseteq \overline{\text{Ran } T^\alpha} \subseteq \overline{\text{Ran } T}$$

so $\overline{\text{Ran } T^\alpha} = \overline{\text{Ran } T}$.

The case $\alpha \leq 1$ is treated in a similar way. ■

Corollary 7.9 *Consider a Hilbert C^* -modules E over a C^* -algebra A and a positive element $T \in \mathcal{R}(E)$. Then we have for every $\alpha \in \mathbb{R}_0^+$ that $\ker T = \ker T^\alpha$.*

This follows because $\ker T = (\text{Ran } T)^\perp$ and $\ker T^\alpha = (\text{Ran } T^\alpha)^\perp$.

An important application of the functional calculus of section 6, can be found in the following definition.

Definition 7.10 *Consider a strictly positive regular operator T in E . Let z be a complex number. Define the function f from \mathbb{R}_0^+ into \mathbb{C}_0 such that $f(t) = t^z$ for every $t \in \mathbb{R}_0^+$. Then we define $T^z = f(T)$, so T^z is an invertible normal regular operator in E .*

Proposition 7.11 *Consider a strictly positive regular operator T in E . Then we have the following calculation rules.*

1. We have for every $s \in \mathbb{R}$ that T^s is a strictly positive regular operator in E .
2. We have for every $s \in \mathbb{R}$ that T^{is} is a unitary element of $\mathcal{L}(E)$.
3. Let s be a real number and z a complex number. Then $(T^s)^z = T^{sz}$.
4. Let n be an integer and z a complex number. Then $(T^z)^n = T^{nz}$.
5. We have for every $z \in \mathbb{C}$ that $(T^z)^* = T^{\bar{z}}$.
6. Consider complex numbers y, z such that $\text{Im } y$ lies between 0 and $\text{Im } z$. Then $D(T^z) \subseteq D(T^y)$.
7. Consider $y, z \in \mathbb{C}$. Then $T^y T^z$ is closable and the closure is equal to T^{y+z} . We have moreover that $D(T^y T^z) = D(T^z) \cap D(T^{y+z})$ and $\text{Ran}(T^y T^z) = \text{Ran}(T^y) \cap \text{Ran}(T^{y+z})$.
8. Consider $y, z \in \mathbb{C}$ such that y and z lie at the same side of the real line. Then $T^y T^z = T^{y+z}$.
9. We have for every $z \in \mathbb{C}$ and $s \in \mathbb{R}$ that $T^{z+is} = T^z T^{is} = T^{is} T^z$.

Corollary 7.12 Consider a positive regular operator T in E which is adjointable invertible. Then

1. We have for every $z \in \mathbb{C}$ with $\text{Re } z \leq 0$ that T^z belongs to $\mathcal{L}(E)$.
2. We have for every $z \in \mathbb{C}$ with $\text{Re } z \geq 0$ that T^z is adjointable invertible.

Proposition 7.13 Consider a strictly positive regular operator T in E . Then we have that the function $\mathbb{R} \rightarrow \mathcal{L}(E) : s \mapsto T^{is}$ is a strongly continuous unitary group homomorphism.

As usual, we can switch between strictly positive elements and selfadjoint elements using the exponential and the logarithm.

Definition 7.14 Consider a normal regular operator T in E . Define the function f from \mathbb{C} into \mathbb{C} such that $f(c) = e^c$ for every $c \in \mathbb{C}$. Then we define $e^T = f(T)$, so e^T is a normal operator in E .

Proposition 1.20 implies immediately the following result.

Corollary 7.15 Consider a selfadjoint regular operator T in E . Then e^T is a strictly positive regular operator in E .

Definition 7.16 Consider a strictly positive regular operator in E . Define the function f from \mathbb{R}_0^+ into \mathbb{R} such that $f(t) = \ln t$ for every $t \in \mathbb{R}_0^+$. Then we define $\ln T = f(T)$, so $\ln T$ is a selfadjoint regular operator in E .

By using cuttings of the plane, we can of course define the logarithm also for a wider class of normal regular operators.

As usual, proposition 6.19 implies the following result.

Result 7.17 We have the following properties :

1. We have for every selfadjoint element $T \in \mathcal{R}(E)$ that $\ln(e^T) = T$.

2. We have for every strictly positive element $T \in \mathcal{R}(E)$ that $e^{\ln T} = T$.

So we see in fact that the functional calculus theory on Hilbert spaces can be rather successfully generalized to a functional calculus theory on Hilbert C^* -modules.

Proposition 7.18 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and let T be an element in $\mathcal{R}(E, F)$. Then we define $|T| = (T^*T)^{\frac{1}{2}}$, so $|T|$ is a positive element in $\mathcal{R}(E)$. We have moreover that $D(T) = D(|T|)$ and that $\langle T(v), T(w) \rangle = \langle |T|(v), |T|(w) \rangle$ for every $v, w \in D(T)$.*

Proof : Because $T^*T = |T|^2$, the set $D(T^*T)$ is a core for T and $|T|$. It is straightforward to check that $\langle T(v), T(w) \rangle = \langle |T|(v), |T|(w) \rangle$ for every $v, w \in D(T^*T)$.

This implies also that $\|T(v)\| = \||T|(v)\|$ for every $v \in D(T^*T)$. Using this equality and the fact that $D(T^*T)$ is a core for both T and $|T|$, it is easy to check that $D(T) = D(|T|)$.

Also the last equality can now be easily deduced. ■

Result 7.19 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and an element $T \in \mathcal{R}(E, F)$. Then $\overline{\text{Ran } |T|} = \overline{\text{Ran } T^*}$ and $\ker |T| = \ker T$*

The result concerning the image follows by applying proposition 4.18 to $|T|$ and T^* . The result concerning the kernels follows from proposition 7.18.

By result 1.19 and the functional calculus rules, we have immediately the following result.

Result 7.20 *Consider a Hilbert C^* -module E over a C^* -algebra A . Let B be a C^* -algebra and π a non-degenerate $*$ -homomorphism from B into $\mathcal{L}(E)$. Then we have for every element T affiliated with B that $\pi(|T|) = |\pi(T)|$.*

In chapter 9 of [7], Lance proved the next result (Remember that the adjoint invertibility of $1 + T^*T$ is equivalent to the fact that $\text{Ran}(1 + T^*T) = E$).

Proposition 7.21 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and $T \in \mathcal{R}(E, F)$. Then $1 + T^*T$ is adjointable invertible, $1 - z_T^* z_T$ is invertible and $(1 + T^*T)^{-1} = 1 - z_T^* z_T$.*

He proved also the next result (this has in fact everything to do with the equivalence of the two approaches to regular operators).

Proposition 7.22 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and $T \in \mathcal{R}(E, F)$. Then*

$$z_T = T(1 + T^*T)^{-\frac{1}{2}} \quad \text{and} \quad T = z_T(1 - z_T^* z_T)^{-\frac{1}{2}}$$

To be more precise, it is proven in [7] that

$$z_T = T((1 + T^*T)^{-1})^{\frac{1}{2}} \quad \text{and} \quad T = z_T((1 - z_T^* z_T)^{\frac{1}{2}})^{-1}$$

but these two results are the same by the results of this section.

If T is a normal, the results above and the functional calculus rules imply that z_T is a function of T .

Corollary 7.23 Consider a Hilbert C^* -module E over a C^* -algebra A and a normal element T in $\mathcal{R}(E)$. Define the continuous function f from \mathbb{C} into \mathbb{C} such that $f(c) = \frac{c}{1+|c|^2}$ for every $c \in \mathbb{C}$. Then we have that $z_T = f(T)$.

Remark 7.24 Define $D = \{c \in \mathbb{C} \text{ such that } |c| < 1\}$. Then we can define the function g from D into \mathbb{C} such that $g(c) = \frac{c}{1-|c|^2}$ for $c \in D$. So g is nothing but the inverse of f . Then we should also have that $T = g(z_T)$

But this needs a more general kind of functional calculus than introduced in section 6. The finite set K appearing in this section should be replaced by a closed set (which in this case would be the unit circle). One should find a general (and easy to check) condition on elements of K (or probably, on the set K) such that proposition 6.5 remains true.

The condition stated in proposition 6.5 is not strong enough. Just look at the case of a normal operator in a Hilbert space which has no eigenvalues in which case K can be any closed subset of the spectrum. In the classical commutative case however, the condition in proposition 6.5 is strong enough because it is equivalent to saying that the range of the function T does not meet K .

8 The Fuglede-Putnam theorem for Hilbert C^* -modules

A well known result in Hilbert space theory is the Fuglede-Putnam theorem. We will prove in this section the Fuglede-Putnam theorem for regular normal operators between Hilbert C^* -modules. The proof is modelled on the proof for the Hilbert space case (see exercices 5 and 8 on page 334 of [4]) but a little bit of care has to be taken.

The next result guarantees that the procedure of cutting off unbounded operators remains a nice (and useful) operation in the Hilbert C^* -module framework.

Lemma 8.1 Consider a Hilbert C^* -module E over a C^* -algebra A and a normal element $T \in \mathcal{R}(E)$. Let G be a subset of \mathbb{C} such that G is compatible with T . Take $f \in K(G)$ and call K the support of f . Define F as the closure of $f(T)E$ in E , so F is a sub Hilbert C^* -module of E . Then we have the following properties.

- We have that $F \subseteq D(T), D(T^*)$, T_F is a normal element in $\mathcal{L}(F)$ and $(T_F)^* = (T^*)_F$.
- The set G is compatible with T_F .
- Consider $g \in C(G)$. Then $F \subseteq D(g(T))$, $g(T)_F$ belongs to $\mathcal{L}(F)$, $\|g(T)_F\| \leq \|g_K\|$ and $g(T)_F = g(T_F)$.

Proof :

- Choose $g \in C(G)$. We have that $|gf| \leq \|g_K\| |f|$. Take $v \in E$. Then we have that $f(T)v$ belongs to $D(g(T))$ and $g(T)(f(T)v) = (gf)(T)v$. So we see that

$$\begin{aligned} \langle g(T)(f(T)v), g(T)(f(T)v) \rangle &= \langle (gf)(T)v, (gf)(T)v \rangle = \langle |gf|^2(T)v, v \rangle \\ &\leq \|g_K\|^2 \langle |f|^2(T)v, v \rangle = \|g_K\|^2 \langle f(T)v, f(T)v \rangle \end{aligned}$$

which implies that $\|g(T)(f(T)v)\| \leq \|g_K\| \|f(T)v\|$.

Hence, using the closedness of $g(T)$, this allows us to conclude that $F \subseteq D(g(T))$ and that $\|g(T)w\| \leq \|g_K\| \|w\|$ for $w \in F$. So $g(T)_F$ is a bounded operator from F into E such that $\|g(T)_F\| \leq \|g_K\|$.

We have for every $v \in D(g(T))$ that $g(T)_F(f(T)v) = g(T)(f(T)v) = f(T)(g(T)v)$ which implies that $g(T)_F(F) \subseteq F$. So $g(T)_F$ is a bounded operator on F .

- Choose $g \in C(G)$. Then we have bounded operators $g(T)_F, \overline{g}(T)_F$ on F . Because $g(T)^* = \overline{g}(T)$, it follows easily that $g(T)_F$ belongs to $\mathcal{L}(F)$ and that $(g(T)_F)^* = \overline{g}(T)_F$.

- We get in particular that $F \subseteq D(T), D(T^*)$, that T_F belongs to $\mathcal{L}(F)$ and that $(T_F)^* = (T^*)_F$. Because $T^*T = TT^*$, this implies also immediately that T_F is normal.

- Define the mapping π from $C_0(G)$ into $\mathcal{L}(F)$ such that $\pi(g) = g(T)_F$ for every $g \in C_0(G)$. Then we get immediately that π is a $*$ -homomorphism.

Furthermore,

$$\begin{aligned} (\pi(C_0(G)) f(F)E)^\perp &= [g(T)f(T)v \mid g \in C_0(G), v \in E] \\ &= [f(T)g(T)v \mid g \in C_0(G), v \in E] = [f(T)w \mid w \in E] = F \end{aligned}$$

which implies that π is non-degenerate.

Take $g \in C(G)$. By lemma 4.2, we know that $\pi(K(G))F$ is a core for $\pi(g)$. We have moreover for every $v \in F$ and $h \in K(G)$ that

$$\pi(g)(\pi(h)v) = \pi(gh)v = (gh)(T)v = g(T)(h(T)v) = g(T)(\pi(h)v) = g(T)_F(\pi(h)v)$$

From this, it follows that $\pi(g) = g(T)_F$.

So we have in particular that $\pi(\iota_G) = T_F$. By proposition 6.17, this implies that T_F is compatible with G . It implies also that π is the functional calculus of T_F . Hence, $g(T_F) = \pi(g) = g(T)_F$ for $g \in C(G)$. ■

It is possible (and not too hard) to prove a similar result if f belongs to $C(G)$, but $g(T)_F$ will then belong to $\mathcal{R}(F)$ (and $D(g(T)_F) = D(g(T)) \cap F$).

We will need the previous lemma in its full generality later on. Let us start of with the following easy implication.

Lemma 8.2 *Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Let r be a strictly positive number, f an element in $K(\mathbb{C})$ such that $f(c) = 0$ for every $c \in \mathbb{C}$ with $|c| \geq r$. Define F as the closure of $f(T)E$ in E . Then we have for every $n \in \mathbb{N}$ and $w \in F$ that w belongs to $D(T^n)$ and $\|T^n(w)\| \leq r^n \|w\|$.*

This lemma follows from the previous lemma applied to the function $g \in C(\mathbb{C})$ defined by $g(c) = c^n$ for $c \in \mathbb{C}$. We know for this function that $g(T) = T^n$ by result 7.1.

Now we will prove a last lemma which is some kind of converse of the previous one.

Lemma 8.3 *Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Let r be a strictly positive number and f an element in $C_b(\mathbb{C})$ such that $f(c) = 1$ for every $c \in \mathbb{C}$ with $|c| \leq r$. Consider an element w in E such that we have for every $n \in \mathbb{N}$ that w belongs to $D(T^n)$ and $\|T^n(w)\| \leq r^n \|w\|$. Then $w = f(T)w$.*

Proof : Define $D = \{c \in \mathbb{C} \mid |c| \leq r\}$.

Take an element $h \in K(\mathbb{C})$ such that h has its support in $\mathbb{C} \setminus D$ and such that $0 \leq h \leq 1$.

Denote the support of h by M , so M is a compact subset of $\mathbb{C} \setminus D$. Then there exists a strictly positive number s such that $s > r$ and such that $|c| > s$ for every $c \in M$.

Take $n \in \mathbb{N}$.

Define the element $g_n \in C(\mathbb{C})$ such that $g_n(c) = c^n$ for every $c \in \mathbb{C}$. Then result 7.1 implies that $T^n = g_n(T)$. So we have by assumption that w belongs to $D(g_n(T))$ and that we have the inequality $\|g_n(T)w\| \leq r^n \|w\|$. (*)

It is clear that $|g_n(c)| > s^n$ for every $c \in M$. By lemma 4.17, this implies the existence of an element $h_n \in K(\mathbb{C})$ such that $0 \leq h_n \leq 1$, $h_n = 1$ on M and $\langle g_n(T)u, \langle g_n(T)u \rangle \geq s^n \langle h_n(T)u, \langle h_n(T)u \rangle$ for every $u \in D(g_n(T))$.

So, using (*), we get that

$$r^n \|w\| \geq \|g_n(T)w\| \geq s^n \|h_n(T)w\| \geq s^n \|h(T)w\|$$

where we used the fact that $h \leq h_n$ in the last inequality.

Hence, we see that $\|h(T)w\| \leq (\frac{r}{s})^n \|w\|$.

Because $\frac{r}{s} < 1$, this implies that $\|h(T)w\| = 0$. So we get that $h(T)w = 0$.

Now take an approximate unit $(e_k)_{k \in K}$ for $C_0(\mathbb{C} \setminus D)$ in $K(\mathbb{C} \setminus D)$. We define for every $k \in K$ the element $d_k \in K(\mathbb{C})$ such that $d_k = 0$ on D and $d_k = e_k$ on $\mathbb{C} \setminus D$. By the first part of the proof, we know that $d_k(T)w = 0$ for every $k \in K$.

Choose $p \in C_0(\mathbb{C})$. Denote the restriction of f and p to $\mathbb{C} \setminus D$ by \tilde{f} and \tilde{p} respectively.

Then $(1 - \tilde{f})\tilde{p}$ belongs to $C_0(\mathbb{C} \setminus D)$ (the first factor takes care of the behaviour in the neighbourhood of the boundary of D , the second factor takes care of the behaviour for large complex numbers).

Because $f = 1$ on D , we have moreover for every $k \in K$ that

$$\|(1 - f)d_k p - (1 - f)p\| = \|(1 - \tilde{f})e_k \tilde{p} - (1 - \tilde{f})\tilde{p}\|.$$

This implies that $((1 - f)d_k p)_{k \in K}$ converges to $(1 - f)p$.

So we see that $((1 - f)d_k)_{k \in K}$ is a bounded net which converges strictly to $1 - f$. Consequently, the net $((1 - f)d_k)(T)_{k \in K}$ converges strongly to $(1 - f)(T)$.

We have for every $k \in K$ that $[(1 - f)d_k](T)w = (1 - f)(T)d_k(T)w = 0$. This implies that $(1 - f)(T)w = 0$, so $f(T)w = w$. ■

First we need the bounded form of the Fuglede-Putnam theorem. The proof of this one can be directly copied from the proof of the Hilbert space form (see e.g. the proof of theorem 6.7 of [4]).

Proposition 8.4 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and U a bounded linear mapping from E into F . Let S be a normal element in $\mathcal{L}(E)$ and T a normal element in $\mathcal{L}(F)$ such that $U S = T U$. Then $U S^* = T^* U$.*

Corollary 8.5 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and U a bounded linear mapping from E into F . Let S be a normal element in $\mathcal{L}(E)$ and T a normal element in $\mathcal{L}(F)$ such that $U S = T U$. Then we have for every $f \in C(\sigma(S) \cup \sigma(T))$ that $U f(S) = f(T) U$.*

Now we will prove the unbounded forms of these results.

Proposition 8.6 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and U a bounded linear mapping from E into F . Let S be a normal element in $\mathcal{R}(E)$ and T a normal element in $\mathcal{R}(F)$ such that $U S \subseteq T U$. Then $U S^* \subseteq T^* U$.*

Proof : We may suppose that $\|U\| \leq 1$. Define for every $m \in \mathbb{N}$ the set $D_m = \{c \in \mathbb{C} \mid |c| \leq m\}$.

Choose $m \in \mathbb{N}$ and take an element $e_m \in K(\mathbb{C})$ such that $0 \leq e_m \leq 1$, such that $e_m = 1$ on D_m and $e_m = 0$ on $\mathbb{C} \setminus D_{m+1}$ (such an element clearly exists).

- We define E_m as the closure of $e_m(S)E$ in E . So E_m is a sub Hilbert C^* -module of E . By lemma 8.1, we know that $E_m \subseteq D(S), D(S^*)$. Denote the restrictions of S and S^* to E_m by S_m and $(S^*)_m$ respectively.

This same lemma implies then moreover that S_m belongs to $\mathcal{L}(E_m)$ and that $(S_m)^* = (S^*)_m$.

- Analogously, we define F_m as the closure of $e_m(T)F$ in F . Then we have that $F_m \subseteq D(T), D(T^*)$ and we denote the restrictions of T and T^* to F_m by T_m and $(T^*)_m$ respectively.

Then we have again that T_m belongs to $\mathcal{L}(F_m)$ and that $(T_m)^* = (T^*)_m$.

Choose $v \in D(S^*)$.

We will further reduce the problem to a bounded one. Therefore take $m \in \mathbb{N}$.

Take $w \in E_m$. Choose $n \in \mathbb{N}$.

By lemma 8.2, we know that w belongs to $D(S^n)$ and that $\|S^n(w)\| \leq (m+1)^n \|w\|$.

Because $US \subseteq TU$, we have also that $US^n \subseteq T^n U$. This implies that $U(w)$ belongs to $D(T^n)$ and that

$$\|T^n(U(w))\| = \|U(S^n(w))\| \leq \|S^n(w)\| \leq (m+1)^n \|w\|.$$

By lemma 8.3, this implies that $U(w) = e_{m+1}(T)U(w)$, so $U(w)$ belongs to F_{m+1} .

Define U_m as the restriction of U to E_m . Then the above argument shows that U_m is a bounded linear operator from E_m into F_{m+1} .

The fact that $US \subseteq TU$ implies then immediately that $U_m S_m = T_{m+1} U_m$. So proposition 8.4 implies that $U_m (S_m)^* = (T_{m+1})^* U_m$. Hence we get that $U_m (S^*)_m = (T^*)_{m+1} U_m$.

We have that $U(e_m(T)v)$ belongs to F_{m+1} , so $U(e_m(T)v)$ belongs to $D(T^*)$. Furthermore,

$$\begin{aligned} T^*(U(e_m(S)v)) &= (T^*)_{m+1}(U_m(e_m(S)v)) = U_m((S^*)_m(e_m(S)v)) \\ &= U(S^*(e_m(S)v)) = U(e_m(S)(S^*(v))) \end{aligned}$$

So we see that $(U(e_m(S)v))_{m=1}^\infty$ converges to $U(v)$ and that $(T^*(U(e_m(S)v)))_{m=1}^\infty$ converges to $U(S^*(v))$. Therefore, the closedness of T^* implies that $U(v)$ belongs to $D(T^*)$ and that $T^*(U(v)) = U(S^*(v))$. ■

Lemma 8.7 Consider Hilbert C^* -modules E, F over a C^* -algebra A and a bounded linear mapping U from E into F . Let B be a C^* -algebra, π a non-degenerate $*$ -homomorphism from B into $\mathcal{L}(E)$ and θ a non-degenerate $*$ -homomorphism from B into $\mathcal{L}(F)$ such that $U\pi(b) = \theta(b)U$ for every $b \in B$. Then $U\pi(T) \subseteq \theta(T)U$ for every $T \in \mathcal{R}(B)$.

Proof : Choose $b \in D(T)$ and $v \in E$.

Now $U(\pi(b)v) = \theta(b)(Uv)$, so $U(\pi(b)v)$ belongs to $D(\theta(T))$ and

$$\theta(T)(U(\pi(b)v)) = \theta(T)(\theta(b)(Uv)) = \theta(T(b))(Uv) = U(\pi(T(b))v).$$

We also have that $\pi(b)v$ belongs to $D(\pi(T))$ and $\pi(T)(\pi(b)v) = \pi(T(b))v$. So we see that $\theta(T)(U(\pi(b)v)) = U(\pi(T)(\pi(b)v))$.

Using the closedness of $\theta(T)$ and the fact that $\langle \pi(b)v \mid b \in D(T), v \in E \rangle$ is a core for $\pi(T)$, we get now easily that $U\pi(T) \subseteq \theta(T)U$. ■

Now we can prove the final form of the Fuglede-Putnam theorem.

Theorem 8.8 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and a bounded linear mapping U from E into F . Let S be a normal element in $\mathcal{R}(E)$ and T a normal element in $\mathcal{R}(F)$ such that $US \subseteq TU$. Consider a subset G of \mathbb{C} which is compatible with both S and T . Then we have for every $f \in C(G)$ that $Uf(S) \subseteq f(T)U$.*

Proof : By proposition 8.6, we know that also $US^* \subseteq T^*U$. This will imply that $U(S^*S) \subseteq (T^*T)U$. In turn, this will imply that $U(1 + S^*S) \subseteq (1 + T^*T)U$. From this, we get that $U(1 + S^*S)^{-1} \subseteq (1 + T^*T)^{-1}U$.

But $(1 + S^*S)^{-1}$ and $(1 + T^*T)^{-1}$ are positive elements in $\mathcal{L}(E)$ and $\mathcal{L}(F)$ respectively. So $U(1 + S^*S)^{-1} = (1 + T^*T)^{-1}U$. Hence, corollary 8.5 implies that $U((1 + S^*S)^{-1})^{\frac{1}{2}} = ((1 + T^*T)^{-1})^{\frac{1}{2}}U$.

This will in turn imply that $US((1 + S^*S)^{-1})^{\frac{1}{2}} \subseteq T((1 + T^*T)^{-1})^{\frac{1}{2}}U$.

So, using proposition 7.22, we get that $Uz_S \subseteq z_TU$ which implies that $Uz_S = z_TU$.

Let us now take a function $g \in C_0(G)$. Then there exist $h \in C_0(\mathbb{C})$ such that $h_G = g$.

Define $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and define the homeomorphism J from \mathbb{C} to D^0 such that $J(c) = \frac{c}{(1+|c|^2)^{\frac{1}{2}}}$ for every $c \in \mathbb{C}$. Then there exists a unique element $k \in C(D)$ such that $k = 0$ on ∂D and such that $k \circ J = h$. By equality 1.19 of [13], we have that $k(z_S) = h(S)$ and $k(z_T) = h(T)$.

Because $Uz_S = z_TU$, we know by corollary 8.5 that $Uk(z_S) = k(z_T)U$. So we get that $Uh(S) = h(T)U$ which implies that $Ug(S) = g(T)U$.

The proposition follows by the previous lemma. ■

Corollary 8.9 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and a unitary operator U from E into F . Let S be a normal element in $\mathcal{R}(E)$ and T a normal element in $\mathcal{R}(F)$ such that $US \subseteq TU$. Consider a subset G of \mathbb{C} which is compatible with both S and T . Then we have for every $f \in C(G)$ that $Uf(S) = f(T)U$.*

Proof : By the previous proposition, we know that $Uf(S) \subseteq f(T)U$.

This same proposition implies also that $U\bar{f}(S) \subseteq \bar{f}(T)U$. So, taking the adjoint of this last inclusion, we get that $U^*f(T) \subseteq f(S)U^*$. In turn, this implies that $f(T)U \subseteq Uf(S)$.

The corollary follows. ■

Corollary 8.10 *Consider a Hilbert C^* -module E over a C^* -algebra A and let S and T be normal elements in $\mathcal{R}(E)$. If $S \subseteq T$, then $S = T$.*

9 Natural left and right multipliers of a regular operator

In this section, we will show that any regular operator has enough well-behaved left and right multipliers.

The functional calculus for normal operators guarantees immediately the existence of well behaved left and right multipliers. Recall the following result from section 4.

Result 9.1 *Consider a Hilbert C^* -module E over a C^* -algebra A and a normal element $T \in \mathcal{R}(E)$. Let G be a subset of \mathbb{C} which is compatible with T and let f be an element in $C(G)$. Consider moreover an element $g \in K(G)$. Then $g(T)$ is a left and right multiplier of $f(T)$ and*

$$g(T) f(T) \subseteq f(T) g(T) = (fg)(T)$$

Result 9.2 *Consider a Hilbert C^* -module E over a C^* -algebra A and a normal element $T \in \mathcal{R}(E)$. Let G be a subset of \mathbb{C} which is compatible with T and let f be an element in $C(G)$. Consider moreover a bounded net $(e_i)_{i \in I}$ in $K(G)$ which converges strictly to 1. Then the following properties hold.*

- *We have for every $v \in D(f(T))$ that $(f(T) e_i(T) v)_{i \in I}$ converges to $f(T)v$.*
- *Consider $v \in E$. Then v belongs to $D(f(T)) \Leftrightarrow$ The net $(f(T) e_i(T) v)_{i \in I}$ is convergent.*

In this section, we will prove an abnormal version of the previous result. We will at the same time be a little bit more general.

Lemma 9.3 *Consider Hilbert C^* -modules E, F, G over a C^* -algebra A . Then we have the following properties :*

- *Consider $T \in \mathcal{R}(F, G)$ and $x \in \mathcal{L}(E, F)$.
Then x is a right multiplier of $T \Leftrightarrow x$ is a right multiplier of $|T|$.*
- *Consider $T \in \mathcal{R}(E, F)$ and $x \in \mathcal{L}(F, G)$.
Then x is a left multiplier of $T \Leftrightarrow x$ is a left multiplier of $|T^*|$.*

The first statement follows immediately from the fact that $D(T) = D(|T|)$ and result 2.3. The second follows from the first by using corollary 2.8.

Lemma 9.4 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and an element $T \in \mathcal{R}(E, F)$. Let $(x_i)_{i \in I}$ be a net in $\mathcal{L}(E)$ which converges strongly to 1 and such that we have for every $i \in I$ that x_i is a left and right multiplier of $|T|$ and that $x_i |T| \subseteq |T| x_i$. Then we have the following properties.*

1. *We have for every $v \in D(T)$ that $(T x_i v)_{i \in I}$ converges to $T(v)$.*
2. *Consider $v \in E$. Then v belongs to $D(T) \Leftrightarrow$ The net $(T x_i v)_{i \in I}$ is convergent.*

Proof : Choose $v \in D(T)$. Then v belongs also to $D(|T|)$.

We have moreover for every $i \in I$ that

$$\|T(x_i v) - T(v)\| = \|T(x_i v - v)\| = \||T|(x_i v - v)\| = \|x_i |T|(v) - |T|(v)\|$$

which guarantees that the net $(T(x_i v))_{i \in I}$ converges to $T(v)$.

The other implication of the second statement follows immediately from the closedness of T . ■

A similar results holds for right multipliers.

Lemma 9.5 Consider Hilbert C^* -modules E, F over a C^* -algebra A and an element $T \in \mathcal{R}(E, F)$. Let $(x_i)_{i \in I}$ be a net in $\mathcal{L}(E)$ which converges strongly* to 1 and such that we have for every $i \in I$ that x_i is a left and right multiplier of $|T^*|$ and that $x_i |T^*| \subseteq |T^*| x_i$. Then we have the following properties.

1. We have for every $v \in D(T)$ that $((x_i \cdot T)v)_{i \in I}$ converges to $T(v)$.
2. Consider $v \in E$. Then v belongs to $D(T) \Leftrightarrow$ The net $((x_i \cdot T)v)_{i \in I}$ is convergent.

Proof : The first statement follows immediately because $(x_i \cdot T)v = x_i T(v)$ for $i \in I$ and $v \in D(T)$.

Take $v \in E$ and $w \in F$ such that $((x_i \cdot T)v)_{i \in I}$ converges to w .

Choose $u \in D(T^*)$. We have for every $i \in I$ that x_i^* is a left and right multiplier of $|T^*|$ and that $x_i^* |T^*| \subseteq |T^*| x_i^*$. We have moreover for every $i \in I$ that

$$\langle (x_i \cdot T)v, u \rangle = \langle v, (x_i \cdot T)^* u \rangle = \langle v, (T^* x_i^*) u \rangle$$

If $i \rightarrow \infty$, the left hand side of this equation converges to $\langle w, u \rangle$, whilst the right hand side converges to $\langle v, T^* u \rangle$ by the previous lemma. So we get that $\langle w, u \rangle = \langle v, T^* u \rangle$.

Because $T^{**} = T$, this implies that $v \in D(T)$ and $T(v) = w$. ■

The next lemma is a key result in this section.

Lemma 9.6 Consider Hilbert C^* -modules E, F over a C^* -algebra A and an element $T \in \mathcal{R}(E, F)$. Let G be a subset of \mathbb{R}^+ which is compatible with both T^*T and TT^* and let f be an element in $C(G)$. Consider $g \in K(G)$ and define C as the closure of $g(T^*T)E$ in E , so C is a sub Hilbert C^* -module of E . Then we have that $C \subseteq D(T f(T^*T)), D(f(TT^*)T)$ and that $(T f(T^*T))v = (f(TT^*)T)v$ for $v \in C$.

Proof : By lemma 8.1, we know that $C \subseteq D(|T|)$ and that $|T|_C$ belongs to $\mathcal{L}(C)$. Because $D(T) = D(|T|)$, we have that $C \subseteq D(T)$. We have moreover that

$$\|T_C(v)\| = \|T(v)\| = \||T|(v)\| \leq \||T|_C\| \|v\|$$

for $v \in C$. So we see that T_C is a bounded A -linear mapping from C into F .

By lemma 8.1, we know also that $C \subseteq D(T^*T)$. The same lemma implies also that $(T^*T)_C$ is a positive element in $\mathcal{L}(C)$.

Because $T(T^*T) = (TT^*)T$, we have immediately that $T_C(T^*T)_C = (TT^*)_C T_C$.

We have also that G is compatible with $(T^*T)_C$ so the Fuglede-Putnam theorem (theorem 8.8) guarantees that $T_C f((T^*T)_C) \subseteq f(TT^*)_C T_C$, so $T_C f((T^*T)_C) = f(TT^*)_C T_C$.

Again, lemma 8.1 gives us that $C \subseteq D(f(T^*T))$, that $f(T^*T)_C$ belongs to $\mathcal{L}(C)$ and that $f(T^*T)_C = f((T^*T)_C)$. Consequently, $T_C f(T^*T)_C = f(TT^*)_C T_C$. The lemma follows. ■

So we have in particular the following result.

Result 9.7 Consider Hilbert C^* -modules E, F over a C^* -algebra A and an element $T \in \mathcal{R}(E, F)$. Let G be a subset of \mathbb{R}^+ which is compatible with both T^*T and TT^* and let f be an element in $C(G)$. Consider moreover $g \in K(G)$ and $v \in E$. Then we have that $g(T^*T)v$ belongs to $D(T f(T^*T)) \cap D(f(TT^*)T)$, that $(fg)(T^*T)$ belongs to $D(T)$ and that

$$(f(TT^*)T)(g(T^*T)v) = (T f(T^*T))(g(T^*T)v) = T((fg)(T^*T)v)$$

A first application of this lemma concerns the abnormal version of result 9.1.

Result 9.8 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and an element $T \in \mathcal{R}(E, F)$. Let G be a subset of \mathbb{R}^+ which is compatible with both T^*T and TT^* and let f be an element in $K(G)$. Then the element $f(T^*T)$ is a right multiplier of T , the element $f(TT^*)$ is a left multiplier of T and*

$$f(TT^*)T \subseteq T f(T^*T)$$

Proof : By result 9.1, we know that $f(T^*T)$ is a right multiplier of $|T|$, so lemma 9.3 implies that $f(T^*T)$ is a right multiplier of T . We get in a similar way that $f(TT^*)$ is a left multiplier of T . Moreover, result 9.7 implies for every $g \in K(G)$ and $v \in E$ that $g(T^*T)v$ belongs to $D(T f(T^*T)) \cap D(f(TT^*)T)$ and that

$$(T f(T^*T))g(T^*T)v = (f(TT^*)T)g(T^*T)v$$

Now the boundedness of $f(TT^*)T$ and $T f(T^*T)$ implies easily the above inclusion. ■

Referring to lemma 9.4, we get also immediately the following result.

Result 9.9 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and an element $T \in \mathcal{R}(E, F)$. Let G be a subset of \mathbb{R}^+ which is compatible with both T^*T and TT^* . Consider moreover a bounded net $(e_i)_{i \in I}$ in $K(G)$ which converges strictly to 1. Then the following properties hold.*

- *We have for every $v \in D(T)$ that $(T e_i(T^*T)v)_{i \in I}$ converges to Tv .*
- *Consider $v \in E$. Then v belongs to $D(T) \Leftrightarrow$ The net $(T e_i(T^*T)v)_{i \in I}$ is convergent.*

In the next part of this section, we want to prove a version of result 9.8 where $f \in C(G)$. We will of course get unbounded operators so a little bit of care has to be taken.

Proposition 9.10 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and an element $T \in \mathcal{R}(E, F)$. Let G be a subset of \mathbb{R}^+ which is compatible with both T^*T and TT^* and let f be an element in $C(G)$. Then $T f(T^*T)$ and $f(TT^*)T$ are closable and we define $T \cdot f(T^*T)$ as the closure of $T f(T^*T)$ and we define $f(TT^*) \cdot T$ as the closure of $f(TT^*)T$. Then we have that $T \cdot f(T^*T) = f(TT^*) \cdot T$.*

Proof : By proposition 7.18, it is clear that $D(T f(T^*T)) = D(|T| f(T^*T))$ and that

$$\|(T f(T^*T))(v)\| = \|(|T| f(T^*T))(v)\|$$

for $v \in D(T f(T^*T))$. Result 4.5 implies moreover that $|T| f(T^*T)$ is closable, so we get also that $T f(T^*T)$ is closable. We denote the closure of $|T| f(T^*T)$ by R and the closure of $T f(T^*T)$ by S . Then the previous remarks imply that $D(R) = D(S)$ and that $\|R(v)\| = \|S(v)\|$ for $v \in D(S)$. (*)

Choose $v \in D(f(TT^*)T)$. Then $v \in D(T)$ and $T(v) \in D(f(TT^*))$.

Take a bounded net $(e_i)_{i \in I}$ in $K(G)$ such that $(e_i)_{i \in I}$ converges strictly to 1.

Fix $j \in I$. Using result 9.7, we get that $e_j(T^*T)v$ belongs to $D(S) \cap D(f(T^*T)T)$ and that $S(e_j(T^*T)v) = (f(TT^*)T)(e_j(T^*T)v)$.

Result 9.8 gives us that $e_j(TT^*)T \subseteq T e_j(T^*T)$, so we get that $T(e_j(T^*T)v) = e_j(TT^*)T(v)$. Because $T(v)$ belongs to $f(TT^*)$, this in turn implies that

$$\begin{aligned} S(e_j(T^*T)v) &= (f(TT^*)T)(e_j(T^*T)v) = f(TT^*)(e_j(TT^*)T(v)) \\ &= e_j(TT^*)(f(TT^*)(T(v))) = e_j(TT^*)(f(TT^*)T(v)). \end{aligned}$$

Hence, we see that $(e_i(T^*T)v)_{i \in I}$ converges to v and that $(S(e_i(T^*T)v))_{i \in I}$ converges to $(f(TT^*)T)(v)$. Therefore, the closedness of S implies that $v \in D(S)$ and $S(v) = (f(TT^*)T)(v)$.

So we have proven that $f(TT^*)T \subseteq S$. This implies immediately that $f(TT^*)T$ is closable.

Define the set $C = \langle g(T^*T)v \mid g \in K(G), v \in E \rangle$. Then result 4.5 and lemma 4.2 imply that C is a core for R . So $(*)$ implies that C is also a core for S . We know moreover that $C \subseteq D(f(TT^*)T)$ by result 9.7. From this all, we conclude that the closure of $f(TT^*)T$ is equal to S . ■

Notice that we have also proven the following result :

Result 9.11 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and an element $T \in \mathcal{R}(E, F)$. Let G be a subset of \mathbb{R}^+ which is compatible with both T^*T and TT^* and let f be an element in $C(G)$. Define the set $C = \langle g(T^*T)v \mid g \in K(G), v \in E \rangle$. Then $C \subseteq D(Tf(T^*T)) \cap D(f(TT^*)T)$ and C is a core for $T \cdot f(T^*T)$.*

So we have also that $D(Tf(T^*T)) \cap D(f(TT^*)T)$ is a core for $T \cdot f(T^*T)$.

Result 9.12 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and an element $T \in \mathcal{R}(E, F)$. Let G be a subset of \mathbb{R}^+ which is compatible with both T^*T and TT^* and let f be an element in $C(G)$. Then $T \cdot f(T^*T) = f(TT^*) \cdot T$ belongs to $\mathcal{R}(E, F)$.*

Proof : By result 9.7, we have that $T \cdot f(T^*T)$ is a densely defined closed A -linear operator from within E into F . We have also immediately that

$$\overline{f(T^*T)T^*} \subseteq (Tf(T^*T))^* = (T \cdot f(T^*T))^*$$

This implies by result 9.7 that $(T \cdot f(T^*T))^*$ is also densely defined. We get moreover that

$$1 + \overline{f(T^*T)T^*T}f(T^*T) \subseteq 1 + (T \cdot f(T^*T))^*(T \cdot f(T^*T))$$

By result 4.5, we know that $\overline{f(T^*T)T^*T}f(T^*T)$ is closable and that its closure P is equal to $(\overline{f} \iota_G f)(T^*T)$. So $P = (|f|^2 \iota_G)(T^*T)$ which implies that P is positive. Using proposition 5.20, this implies that $\text{Ran}(1 + P) = E$.

So we get that $1 + \overline{f(T^*T)T^*T}f(T^*T)$ has dense range, which in turn implies that the element $1 + (T \cdot f(T^*T))^*(T \cdot f(T^*T))$ has dense range. Hence, we get by definition that $T \cdot f(T^*T)$ belongs to $\mathcal{R}(E, F)$. ■

Result 9.13 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and an element $T \in \mathcal{R}(E, F)$. Let G be a subset of \mathbb{R}^+ which is compatible with both T^*T and TT^* and let f be an element in $C(G)$. Consider $g \in K(G)$. Then we have the following properties.*

- The element $g(T^*T)$ is a right multiplier of $T \cdot f(T^*T)$, $(fg)(T^*T)$ is a right multiplier of T .
- The element $g(TT^*)$ is a left multiplier of $T \cdot f(T^*T)$, $(gf)(TT^*)$ is a right multiplier of T .
- We have the following chain of equalities

$$(T \cdot f(T^*T)) \cdot g(T^*T) = T \cdot (fg)(T^*T) = (gf)(TT^*) \cdot T = g(TT^*) \cdot (T \cdot f(T^*T))$$

Proof :

- The first statement and the first equality in the chain above follow easily from results 9.1 and 9.8.
- By result 9.8, we know that $(fg)(TT^*)$ is a right multiplier of T . We have moreover for every $v \in D(f(TT^*)T)$ that

$$g(TT^*)(f(TT^*)T)(v) = (gf)(TT^*)T(v) = ((gf)(TT^*) \cdot T)(v)$$

This implies for every $v \in D(f(TT^*) \cdot T)$ that $g(TT^*)(f(TT^*) \cdot T)(v) = ((gf)(TT^*) \cdot T)(v)$.

By definition, this gives us that $g(TT^*)$ is a left multiplier of $f(TT^*) \cdot T$ and that $g(TT^*) \cdot (f(TT^*) \cdot T) = (gf)(TT^*) \cdot T$.

Now result 9.8 joins the two equalities. ■

This allows now to prove easily the following result.

Proposition 9.14 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and an element $T \in \mathcal{R}(E, F)$. Let G be a subset of \mathbb{R}^+ which is compatible with both T^*T and TT^* and let f be an element in $C(G)$. Then*

$$(T \cdot f(T^*T))^* = (f(TT^*) \cdot T)^* = \overline{f}(T^*T) \cdot T^* = T^* \cdot \overline{f}(TT^*)$$

Proof : We have that

$$\overline{f}(T^*T)T^* \subseteq (Tf(T^*T))^* = (T \cdot f(T^*T))^*$$

which implies that $\overline{f}(T^*T) \cdot T^* \subseteq (T \cdot f(T^*T))^*$.

Choose $v \in D((T \cdot f(T^*T))^*)$.

Take a bounded net $(e_i)_{i \in I}$ in $K(G)$ which converges strictly to 1.

Take $j \in I$. By the previous lemma, we know that $e_j(TT^*)v$ belongs to $D(T^* \cdot \overline{f}(TT^*))$, that $(\overline{f}e_j)(TT^*)$ is a right multiplier of T^* and that

$$(T^* \cdot \overline{f}(TT^*))(e_j(TT^*)v) = (T^* \cdot (\overline{f}e_j)(TT^*))(v) \quad (a)$$

Because $(\overline{f}e_j)(TT^*)$ is a right multiplier of T^* , we know that $(f\overline{e_j})(TT^*)$ is a left multiplier of T and that

$$T^* \cdot (\overline{f}e_j)(TT^*) = ((f\overline{e_j})(TT^*) \cdot T)^* = (T \cdot (f\overline{e_j})(T^*T))^* \quad (b)$$

Again, we know by the previous result that $\overline{e_j}(T^*T)$ is a right multiplier of $T \cdot f(T^*T)$ and that

$$T \cdot (f\overline{e_j})(T^*T) = (T \cdot f(T^*T)) \cdot \overline{e_j}(T^*T)$$

This implies that $e_j(T^*T)$ is a left multiplier of $(T \cdot f(T^*T))^*$ and that

$$e_j(T^*T) \cdot (T \cdot f(T^*T))^* = ((T \cdot f(T^*T)) \cdot \overline{e_j}(T^*T))^* = (T \cdot (f\overline{e_j})(T^*T))^* = T^* \cdot (\overline{f}e_j)(TT^*)$$

where we used (b) in the last equality. So, using (a), we get that

$$(T^* \cdot \overline{f}(TT^*))(e_j(TT^*)v) = (e_j(T^*T) \cdot (T \cdot f(T^*T))^*)(v) = e_j(T^*T) (T \cdot f(T^*T))^*(v)$$

Hence, we see that $(e_i(TT^*)v)_{i \in I}$ converges to v and that $((T^* \cdot \overline{f}(TT^*))(e_i(TT^*)v))_{i \in I}$ converges to $(T \cdot f(T^*T))^*(v)$. Consequently, the closedness of $T^* \cdot \overline{f}(TT^*)$ implies that v belongs to $D(T^* \cdot \overline{f}(TT^*))$. ■

We will use notation 4.6 in the next propositions.

Proposition 9.15 Consider Hilbert C^* -modules E, F over a C^* -algebra A and an element $T \in \mathcal{R}(E, F)$. Let G be a subset of \mathbb{R}^+ which is compatible with both T^*T and TT^* and let f be an element in $C(G)$. Then

$$(T \cdot f(T^*T))^*(T \cdot f(T^*T)) = (T^*T) \cdot (|f|^2(T^*T)) \quad \text{and} \quad (T \cdot f(T^*T))(T \cdot f(T^*T))^* = (TT^*) \cdot (|f|^2(TT^*))$$

Proof : We have that

$$(T \cdot f(T^*T))^*(T \cdot f(T^*T)) = (f(T^*T) \cdot T^*)(T \cdot f(T^*T)) \supseteq f(T^*T)T^*Tf(T^*T)$$

By result 4.5, we know that $f(T^*T)T^*Tf(T^*T)$ is closable and that the closure is equal to $(\iota_G |f|^2)(T^*T)$. Hence, the previous inclusion implies that

$$(T \cdot f(T^*T))^*(T \cdot f(T^*T)) \supseteq (\iota_G |f|^2)(T^*T)$$

But both operators are selfadjoint, so the inclusion must be an equality. Hence, result 4.5 implies that

$$(T \cdot f(T^*T))^*(T \cdot f(T^*T)) = (\iota_G |f|^2)(T^*T) = T^*T \cdot |f|^2(T^*T)$$

Using the previous proposition, the second equality can be brought in the same form as the first one (with the role of T and T^* interchanged). ■

Using proposition 6.19 and result 4.5, this implies the following result.

Corollary 9.16 Consider Hilbert C^* -modules E, F over a C^* -algebra A and an element $T \in \mathcal{R}(E, F)$. Let G be a subset of \mathbb{R}^+ which is compatible with both T^*T and TT^* and let f be an element in $C(G)$. Then

$$|T \cdot f(T^*T)| = |T| \cdot |f|(T^*T) \quad \text{and} \quad |(T \cdot f(T^*T))^*| = |T^*| \cdot |f|(TT^*)$$

Combining this result with lemma 9.3, we get the following one.

Result 9.17 Consider Hilbert C^* -modules E, F over a C^* -algebra A and an element $T \in \mathcal{R}(E, F)$. Let G be a subset of \mathbb{R}^+ which is compatible with both T^*T and TT^* and let f be an element in $C(G)$. Consider a bounded net $(e_i)_{i \in I}$ in $K(G)$ which converges strictly to 1. Then the following holds.

1. We have for every $v \in D(T \cdot f(T^*T))$ that $((T \cdot f(T^*T))e_i(T^*T)v)_{i \in I}$ converges to $(T \cdot f(T^*T))v$
2. Consider $v \in E$. Then v belongs to $D(T \cdot f(T^*T)) \Leftrightarrow$ The net $((T \cdot f(T^*T))e_i(T^*T)v)_{i \in I}$ is convergent.

It is possible to get some further information about the domains by the following result.

Result 9.18 Consider Hilbert C^* -modules E, F over a C^* -algebra A and an element $T \in \mathcal{R}(E, F)$. Let G be a subset of \mathbb{R}^+ which is compatible with both T^*T and TT^* . Then the following holds.

- Consider $f \in C(G)$. Then we have the following properties.

1. $D(Tf(T^*T)) = D(|T|f(T^*T)) = D(T \cdot f(T^*T)) \cap D(f(T^*T)) = D(|T| \cdot f(T^*T)) \cap D(f(T^*T))$
2. $D(T \cdot f(T^*T)) = D(|T| \cdot f(T^*T))$

- Consider $f, g \in C(G)$. Then we have for every $v \in D(T \cdot f(T^*T))$ and $w \in D(T \cdot g(T^*T))$ that

$$\langle (T \cdot f(T^*T))(v), (T \cdot g(T^*T))(w) \rangle = \langle (|T| \cdot f(T^*T))(v), (|T| \cdot g(T^*T))(w) \rangle$$

Proof :

- Choose $f \in C(G)$. Then proposition 7.18 implies immediately that $D(T f(T^*T)) = D(|T| f(T^*T))$ and that $\|(T f(T^*T))(v)\| = \|(|T| f(T^*T))(v)\|$ for $v \in D(T f(T^*T))$. This implies immediately that $D(T \cdot f(T^*T)) = D(|T| \cdot f(T^*T))$ and that $\|(T \cdot f(T^*T))(v)\| = \|(|T| \cdot f(T^*T))(v)\|$ for $v \in D(T \cdot f(T^*T))$.

Result 4.7 gives us that $D(|T| f(T^*T)) = D(|T| \cdot f(T^*T)) \cap D(f(T^*T))$.

- This follows now easily from the results in the first part of the proof and proposition 7.18. ■

Remark 9.19 We give quickly an example how this can be useful. Therefore suppose that f is an element in $C(G)$ such that there exists a positive number M satisfying $|f(c)| \leq M$ and $\sqrt{c}|f(c)| \leq M$ for $c \in \sigma(T^*T) \cap G$. Then $D(|T| \cdot f(T^*T)) \cap D(f(T^*T)) = E$.

So in this case $T \cdot f(T^*T) = T f(T^*T) \in \mathcal{L}(E, F)$.

A situation where the above terminology applies is the connection between T and its z -transform z_T . Consider Hilbert C^* -module E, F over a C^* -algebra A and $T \in \mathcal{R}(E, F)$. Proposition 7.22 implies immediately the following results.

- $z_T = T \cdot (1 + T^*T)^{-\frac{1}{2}} = (1 + TT^*)^{-\frac{1}{2}} \cdot T$
- $T = z_T \cdot (1 - z_T^* z_T)^{-\frac{1}{2}} = (1 - z_T z_T^*)^{-\frac{1}{2}} \cdot z_T$

10 C^* -algebras of adjointable operators on Hilbert C^* -modules

In this section, we will fix a Hilbert C^* -module E over a C^* -algebra. At the same time, we will consider a non-degenerate sub- C^* -algebra B of $\mathcal{L}(E)$. We will look at an embedding of $\mathcal{R}(B)$ into $\mathcal{R}(E)$.

Concerning the multiplier algebra, we have the following well known result:

$$M(B) = \{ x \in \mathcal{L}(E) \mid \text{We have for every } b \in B \text{ that } xb \text{ and } bx \text{ belong to } B \}$$

As pointed out in [13] for Hilbert spaces, we can also embed $\mathcal{R}(B)$ in $\mathcal{R}(E)$.

Definition 10.1 Call π the inclusion of B into $\mathcal{L}(E)$, then π is a non-degenerate $*$ -homomorphism from B into $\mathcal{L}(E)$. Let T be an element affiliated to B . Then we define $\tilde{T} = \pi(T)$, so \tilde{T} is a regular operator in E .

Because π is injective, we know immediately that the mapping $\mathcal{R}(B) \rightarrow \mathcal{R}(E) : T \mapsto \tilde{T}$ is injective.

We have also immediately that $\tilde{x} = x$ for every $x \in M(B)$.

So we get the following determining property of the embedding :

Result 10.2 *We have that $z_T = z_{\tilde{T}}$ for every element T affiliated with B .*

Because this embedding arises from an injective non-degenerate $*$ -homomorphism, most of the properties of this embedding follow from the theory of injective non-degenerate $*$ -homomorphisms. We will now prove some extra results.

Looking at theorem 1.9, we have immediately the following result :

Result 10.3 *Consider an element T affiliated to B . Then*

- *We have for every $x \in D(T)$ and $v \in E$ that $xv \in D(\tilde{T})$ and $\tilde{T}(xv) = T(x)v$.*
- *$D(T)E$ is a core for \tilde{T}*

Let D be a core for T and K a dense subspace of E . Then DK is a core for \tilde{T} .

We have also some kind of converse of this result.

Proposition 10.4 *Consider an element T affiliated with B and an element x in B . Then*

- *If x belongs to $D(T)$, then x is a right multiplier of \tilde{T} and $\tilde{T} \cdot x = T(x)$.*
- *We have that x belongs to $D(T) \Leftrightarrow x$ is a right multiplier of \tilde{T} and $\tilde{T} \cdot x$ belongs to B .*

Proof : The first statement follows immediately from the previous result.

Choose $x \in B$ such that x is a right multiplier of \tilde{T} and $\tilde{T} \cdot x$ belongs to B .

Take $y \in D(T^*)$. Choose $v, w \in E$.

Because x is a right multiplier of \tilde{T} , we get that xw belongs to $D(\tilde{T})$ and $\tilde{T}(xw) = (\tilde{T} \cdot x)w$. By results 1.19 and 10.3, we have also that yv belongs to $D(\tilde{T}^*)$ and that $\tilde{T}^*(yv) = T^*(y)v$. Hence,

$$\langle x^* T^*(y) v, w \rangle = \langle T^*(y) v, xw \rangle = \langle \tilde{T}^*(yv), xw \rangle = \langle yv, \tilde{T}(xw) \rangle = \langle yv, (\tilde{T} \cdot x)w \rangle = \langle (\tilde{T} \cdot x)^* yv, w \rangle$$

Consequently, $x^* T^*(y) = (\tilde{T} \cdot x)^* y$.

Because $T = T^{**}$, this implies that $x \in D(T)$ and $T(x) = \tilde{T} \cdot x$. ■

Corollary 10.5 *Consider an element T affiliated with B and x an element in $M(B)$. Then*

- *If x is a right multiplier of T , then x is a right multiplier of \tilde{T} and $\tilde{T} \cdot x = T \cdot x$.*
- *We have that x is a right multiplier of $T \Leftrightarrow x$ is a right multiplier of \tilde{T} and $\tilde{T} \cdot x$ belongs to $M(B)$.*

Proof :

- Suppose that x is a right multiplier of T .

Choose $b \in B$ and $v \in E$. Then xb belongs to $D(T)$, so the previous proposition implies that xb is right multiplier of \tilde{T} and $\tilde{T} \cdot (xb) = T(xb) = (T \cdot x)b$. So we get that xbv belongs to $D(\tilde{T})$ and $\tilde{T}(xbv) = (T \cdot x)bv$.

Because BE is dense in E and \tilde{T} is closed, this implies easily for every $w \in E$ that xw belongs to \tilde{T} and that $\tilde{T}(xw) = (T \cdot x)w$. This implies that x is a right multiplier of \tilde{T} and $\tilde{T} \cdot x = T \cdot x$.

We have in particular that $\tilde{T} \cdot x$ belongs to $M(B)$.

- Suppose that x is right multiplier of \tilde{T} and that $\tilde{T} \cdot x$ belongs to $M(B)$.
Choose $b \in B$. Then xb is a right multiplier of \tilde{T} and $\tilde{T} \cdot (xb) = (\tilde{T} \cdot x)b$ which clearly belongs to B .
Hence, the previous proposition implies that $xb \in D(T)$ and $T(xb) = \tilde{T} \cdot (xb) = (\tilde{T} \cdot x)b$.
So we get that x is a right multiplier of T and $T \cdot x = \tilde{T} \cdot x$.

■

By using this result and corollary 2.8 , we have also the following result.

Corollary 10.6 *Consider an element T affiliated with B and an element x in $M(B)$. Then*

- *If x is a left multiplier of T , then x is a left multiplier of \tilde{T} and $x \cdot \tilde{T} = x \cdot T$.*
- *We have that x is a left multiplier of $T \Leftrightarrow x$ is a left multiplier of \tilde{T} and $x \cdot \tilde{T}$ belongs to $M(B)$.*

Looking at example 4 of [13], we have also the following result.

Result 10.7 *Consider a regular operator T in E . Then there exists an element S affiliated with B such that $\tilde{S} = T \Leftrightarrow$*

1. *z_T belongs to $M(B)$*
2. *$(1 - z_T^* z_T)^{\frac{1}{2}} B$ is dense in B*

If there exists such an S , we have immediately that $z_T = z_S$, so z_T will certainly satisfy the two mentioned conditions.

If z_T satisfies these two conditions, there exists an element S affiliated with B such that $z_S = z_T$. So we have that $z_{\tilde{S}} = z_S = z_T$ which implies that $\tilde{S} = T$.

This implies immediately the following result.

Proposition 10.8 *Consider a Hilbert C^* -module over E over a C^* -algebra A . Then the mapping $\mathcal{R}(\mathcal{K}(E)) \rightarrow \mathcal{R}(E) : T \mapsto \tilde{T}$ is a bijection.*

Remember that even for injective non-degenerate $*$ -homomorphisms, the invertibility of an element and its image where not equivalent. It is however true for the bijection between $\mathcal{R}(\mathcal{K}(E))$ and $\mathcal{R}(E)$.

Lemma 10.9 *Consider a Hilbert C^* -module over E over a C^* -algebra A . Let T be an element affiliated with $\mathcal{K}(E)$ and $v \in D(\tilde{T})$. Then we have for every $w \in E$ that $\theta_{v,w}$ belongs to $D(T)$ and $T(\theta_{v,w}) = \theta_{\tilde{T}(v),w}$.*

Proof : Take $w \in E$. Then we have for every $u \in E$ that $\theta_{v,w}(u) = v \langle u, w \rangle$, which implies that $\theta_{v,w}(u)$ belongs to $D(\tilde{T})$ and $\tilde{T}(\theta_{v,w}(u)) = \tilde{T}(v) \langle u, w \rangle = \theta_{\tilde{T}(v),w}(u)$.

So we see that $\theta_{v,w}$ is a right multiplier of \tilde{T} and $\tilde{T} \cdot \theta_{v,w} = \theta_{\tilde{T}(v),w}$. By proposition 10.4, this implies that $\theta_{v,w}$ belongs to $D(T)$ and $T(\theta_{v,w}) = \theta_{\tilde{T}(v),w}$. ■

So we get easily the following result.

Proposition 10.10 *Consider a Hilbert C^* -module over E over a C^* -algebra A and T an element affiliated with $\mathcal{K}(E)$. Then T is invertible $\Leftrightarrow \tilde{T}$ is invertible.*

Corollary 10.11 *Consider a Hilbert C^* -module over E over a C^* -algebra A and T a normal element affiliated with $\mathcal{K}(E)$. Let G be an almost closed subset of \mathbb{C} . Then G is compatible with $T \Leftrightarrow G$ is compatible with \tilde{T} .*

11 Representing Hilbert C^* -modules on Hilbert spaces

Sometimes it is useful to represent Hilbert C^* -modules on Hilbert spaces. In this section, we describe a natural way to do so and show that this procedure is well behaved.

Notation 11.1 Consider a Hilbert C^* -module E over a C^* -algebra A and $\omega \in A_+^*$. As in the case of positive functionals on a C^* -algebra, there is a natural way to construct a Hilbert space E_ω together with a bounded linear map $E \rightarrow E_\omega : v \mapsto \bar{v}$ such that

1. \bar{E} is dense in E_ω .
2. We have for $v, w \in E$ that $\langle \bar{v}, \bar{w} \rangle = \omega(\langle v, w \rangle)$.

It is clear that $\|\bar{v}\| \leq \|\omega\| \|v\|$ for every $v \in E$.

In a next step, we want to represent operators on E by operators on E_ω .

Notation 11.2 Consider Hilbert C^* -modules E, F over a C^* -algebra A and $\omega \in A_+^*$.

Let T be a linear map from E into F such that there exists a positive number M such that $\langle Tv, Tv \rangle \leq M \langle v, v \rangle$ for every $v \in E$. Then there exists a unique bounded linear map T_ω from E_ω into F_ω such that $T_\omega \bar{v} = \overline{Tv}$ for $v \in E$. It is clear that $\|T_\omega\| \leq M^{\frac{1}{2}}$.

This definition is in particular applicable if T is an bounded A -linear map T (e.g. if $T \in \mathcal{L}(E, F)$) in which case $\langle Tv, Tv \rangle \leq \|T\|^2 \langle v, v \rangle$ for every $v \in E$. We have in this case that $\|T_\omega\| \leq \|T\|$.

For the rest of this section, we will only be interested in the case that T belongs to $\mathcal{L}(E, F)$.

The following properties are straightforward to prove.

Result 11.3 Consider Hilbert C^* -modules E, F over a C^* -algebra A and $\omega \in A_+^*$. Then

1. The mapping $\mathcal{L}(E, F) \rightarrow \mathcal{L}(E_\omega, F_\omega)$ is linear
2. We have for every $T \in \mathcal{L}(E, F)$ that $(T_\omega)^* = (T^*)_\omega$
3. The mapping $\mathcal{L}(E, F) \rightarrow \mathcal{L}(E_\omega, F_\omega)$ is bounded and has norm ≤ 1 .
4. The mapping $\mathcal{L}(E, F) \rightarrow \mathcal{L}(E_\omega, F_\omega)$ is strongly continuous on bounded sets.
5. The mapping $\mathcal{L}(E, F) \rightarrow \mathcal{L}(E_\omega, F_\omega)$ is strongly* continuous on bounded sets.

Result 11.4 Consider Hilbert C^* -modules E, F, G over a C^* -algebra A and $\omega \in A_+^*$. Then we have for every $S \in \mathcal{L}(E, F)$, $T \in \mathcal{L}(F, G)$ that $(TS)_\omega = T_\omega S_\omega$.

Let us now consider the case where T is regular operator.

Definition 11.5 Consider Hilbert C^* -modules E, F over a C^* -algebra A and $\omega \in A_+^*$. Let T be an element in $\mathcal{R}(E, F)$. Then there exists a unique densely defined closed linear map T_ω from within E_ω into F_ω such that $\overline{D(T)}$ is a core for T_ω and $T_\omega \bar{v} = \overline{Tv}$ for $v \in E$.

It is easy to see that $\langle \overline{Tv}, \bar{w} \rangle = \langle \bar{v}, \overline{T^*w} \rangle$ for $v \in D(T)$, $w \in D(T^*)$. Because $\overline{D(T)}$ is dense in F_ω , this equality implies easily the existence of the mapping T_ω above.

This definition implies also easily that \overline{D} is a core for T_ω if D is a core for $D(T)$.

In a following proposition, we prove another expected characterization of T_ω .

Proposition 11.6 Consider Hilbert C^* -modules E, F over a C^* -algebra A and $\omega \in A_+^*$. Let T be an element in $\mathcal{R}(E, F)$. Then $(z_T)_\omega = z_{T_\omega}$.

Proof : Because $\|z_T\| \leq 1$, we know that $\|(z_T)_\omega\| \leq 1$. We have moreover that $1 - (z_T)_\omega^*(z_T)_\omega = (1 - z_T^*z_T)_\omega$. Because the mapping $\mathcal{L}(E) \rightarrow \mathcal{B}(E_\omega) : S \rightarrow S_\omega$ is a $*$ -homomorphism, this implies that $((1 - z_T^*z_T)_\omega)^{\frac{1}{2}} = ((1 - z_T^*z_T)^{\frac{1}{2}})_\omega$, so we see that $(1 - (z_T)_\omega^*(z_T)_\omega)^{\frac{1}{2}} = ((1 - z_T^*z_T)^{\frac{1}{2}})_\omega$. (*)

Hence, we get that $(1 - (z_T)_\omega^*(z_T)_\omega)^{\frac{1}{2}} \overline{E} = ((1 - z_T^*z_T)^{\frac{1}{2}} E)^-$. Because $(1 - z_T^*z_T)^{\frac{1}{2}} E$ is dense in E , this implies that $(1 - (z_T)_\omega^*(z_T)_\omega)^{\frac{1}{2}} E_\omega$ is dense in E_ω .

So there exists a densely defined closed linear operator S from within E_ω into F_ω such that $z_S = (z_T)_\omega$. Because \overline{E} is dense in E_ω , we have that $(1 - z_S^*z_S)^{\frac{1}{2}} \overline{E}$ is a core for S . Now (*) implies that

$$(1 - z_S^*z_S)^{\frac{1}{2}} \overline{E} = ((1 - z_T^*z_T)^{\frac{1}{2}})_\omega \overline{E} = ((1 - z_T^*z_T)^{\frac{1}{2}} E)^- = \overline{D(T)}$$

which implies that $\overline{D(T)}$ is a core for S .

Choose $v \in D(T)$. Then there exist $w \in E$ such that $v = (1 - z_T^*z_T)^{\frac{1}{2}} w$. So $Tv = z_T w$.

By (*), we have then also that $\overline{v} = ((1 - z_T^*z_T)^{\frac{1}{2}})_\omega \overline{w} = (1 - (z_T)_\omega^*(z_T)_\omega)^{\frac{1}{2}} \overline{w} = (1 - z_S^*z_S)^{\frac{1}{2}} \overline{w}$.

This implies that $S\overline{v} = z_S \overline{w} = (z_T)_\omega \overline{w} = \overline{z_T w} = \overline{Tv} = T_\omega \overline{v}$.

Therefore we get that $S = T_\omega$. So $(z_T)_\omega = z_S = z_{T_\omega}$. ■

Because of this property, the following two results are very easy to prove.

Result 11.7 Consider Hilbert C^* -modules E, F over a C^* -algebra A and $\omega \in A_+^*$. Let T be an element in $\mathcal{R}(E, F)$. Then we have that $(T_\omega)^* = (T^*)_\omega$.

Result 11.8 Consider a Hilbert C^* -module E over a C^* -algebra A and $\omega \in A_+^*$. Let T be an element in $\mathcal{R}(E)$.

1. If T is normal, then T_ω is normal.
2. If T is selfadjoint, then T_ω is selfadjoint.
3. If T is positive, then T_ω is positive.
4. If T is strictly positive, then T_ω is strictly positive.

There is another useful way to get to T_ω for $T \in \mathcal{R}(E)$, using the bijection $\mathcal{R}(\mathcal{K}(E)) \rightarrow \mathcal{R}(E) : S \mapsto \tilde{S}$.

Result 11.9 Consider a Hilbert C^* -module E over a C^* -algebra A and $\omega \in A_+^*$. Define the function π from $\mathcal{K}(E)$ into $\mathcal{K}(E_\omega)$ such that $\pi(x) = x_\omega$ for every $x \in \mathcal{K}(E)$. Then π is a non-degenerate $*$ -homomorphism of $\mathcal{K}(E)$ into $\mathcal{K}(E_\omega)$ such that $\pi(x) = x_\omega$ for $x \in \mathcal{L}(E)$.

Let T be an element in $\mathcal{R}(E)$. So there exists a unique element $S \in \mathcal{R}(\mathcal{K}(E))$ such that $\tilde{S} = T$. Then we have that $\pi(S) = T_\omega$.

This follows immediately from the fact that $z_{\pi(S)} = \pi(z_S) = (z_S)_\omega = (z_T)_\omega = z_{T_\omega}$.

We can use this result to prove easily the following one.

Proposition 11.10 Consider a Hilbert C^* -module E over a C^* -algebra A and $\omega \in A_+^*$. Let T be a normal element in $\mathcal{R}(E)$. Let G be a subset of \mathbb{C} which is compatible with T . Then we have the following properties.

- The set G is compatible with T_ω .
- We have for every $f \in C(G)$ that $f(T_\omega) = f(T)_\omega$.

Proof : Define the non-degenerate $*$ -homomorphism π from $\mathcal{K}(E)$ into $\mathcal{K}(E_\omega)$ such that $\pi(x) = x_\omega$ for every $x \in \mathcal{K}(E)$.

Take $S \in \mathcal{R}(\mathcal{K}(E))$ such that $\tilde{S} = T$. Then the previous result implies that $\pi(S) = T_\omega$.

By the remark after proposition 1.20 and corollary 10.11, we know that S is normal and that G is compatible with S . This implies that G is compatible with $\pi(S)$. So G is compatible with T_ω .

Take $f \in C(G)$. By proposition 6.17, we have that $\widetilde{f(S)} = f(\tilde{S}) = f(T)$. So the previous result implies that $f(T)_\omega = \pi(f(S))$. Hence, $f(T)_\omega = \pi(f(S)) = f(\pi(S)) = f(T_\omega)$. ■

Using result 11.9, also the next results are easy consequences of already known results.

Result 11.11 Consider Hilbert C^* -modules E, F over a C^* -algebra A and $\omega \in A_+^*$. Let T be an element in $\mathcal{R}(E, F)$. Then we have that $(T_\omega)^*(T_\omega) = (T^*T)_\omega$ and $(T_\omega)(T_\omega)^* = (TT^*)_\omega$.

Result 11.12 Consider Hilbert C^* -modules E, F over a C^* -algebra A and $\omega \in A_+^*$. Let T be an element in $\mathcal{R}(E, F)$. Then

- If T is invertible, then T_ω is invertible and $(T_\omega)^{-1} = (T^{-1})_\omega$.
- If T is adjointable invertible, then T_ω is bounded invertible.

Corollary 11.13 Consider a Hilbert C^* -module E over a C^* -algebra A and $\omega \in A_+^*$. Let T be an element in $\mathcal{R}(E)$. Then $\sigma(T_\omega) \subseteq \sigma(T)$.

Proposition 11.10 implies immediately the following result.

Result 11.14 Consider a Hilbert C^* -module E over a C^* -algebra A and $\omega \in A_+^*$. Let T be a normal element in $\mathcal{R}(E)$. Then we have the following properties.

- We have for every $n \in \mathbb{N} \cup \{0\}$ that $(T_\omega)^n = (T^n)_\omega$.
- If T is invertible, then we have for every $n \in \mathbb{Z}$ that $(T_\omega)^n = (T^n)_\omega$.
- If T is positive, then we have for every $s \in \mathbb{R}^+$ that $(T_\omega)^s = (T^s)_\omega$.
- If T is strictly positive, then we have for every $z \in \mathbb{C}$ that $(T_\omega)^z = (T^z)_\omega$.

The proof of the following lemma is straightforward.

Lemma 11.15 Consider two Hilbert C^* -modules E, F over a C^* -algebra A and let K be a subset of A_+^* which is separating for A . Consider elements S, T in $\mathcal{L}(E, F)$. Then $S = T \Leftrightarrow$ We have for every $\omega \in K$ that $S_\omega = T_\omega$.

Combining this lemma with proposition 11.6, we get immediately the following result.

Proposition 11.16 *Consider two Hilbert C^* -modules E, F over a C^* -algebra A and let K be a subset of A_+^* which is separating for A . Consider elements S, T in $\mathcal{R}(E, F)$. Then $S = T \Leftrightarrow$ We have for every $\omega \in K$ that $S_\omega = T_\omega$.*

Combining this proposition with results 11.7 and 11.11, we get easily the following result.

Corollary 11.17 *Consider a Hilbert C^* -module E over a C^* -algebra A and let K be a subset of A_+^* which is separating for A . Consider an element T in $\mathcal{R}(E)$. Then we have the following properties.*

1. T is normal \Leftrightarrow We have for every $\omega \in K$ that T_ω is normal.
2. T is selfadjoint \Leftrightarrow We have for every $\omega \in K$ that T_ω is selfadjoint.
3. T is positive \Leftrightarrow We have for every $\omega \in K$ that T_ω is positive.

This proposition can then also be used to prove easily the following result.

Proposition 11.18 *Consider a Hilbert C^* -module E over a C^* -algebra A and let S and T be strictly positive elements in $\mathcal{R}(E)$. Then $S = T \Leftrightarrow$ We have for every $t \in \mathbb{R}$ that $S^{it} = T^{it}$.*

In [5], we will use this Hilbert space procedure in the proof of proposition 12.23. This was in fact the main reason to include this section.

12 Commuting normal operators

In this section, we look at an obvious notion of commuting normal regular operators and introduce the basic properties.

Let us start with a familiar definition of strongly commuting operators.

Definition 12.1 *Consider a Hilbert C^* -module E over a C^* -algebra A and let S, T be normal elements in $\mathcal{R}(E)$. Then we say that S (strongly) commutes with $T \Leftrightarrow$ We have for every $f, g \in C_0(\mathbb{C})$ that $f(S)g(T) = g(T)f(S)$.*

It is clear that we get a symmetric definition in this way.

Result 12.2 *Consider a Hilbert C^* -module E over a C^* -algebra A and let S, T be normal elements in $\mathcal{R}(E)$. Let F, G be subsets of \mathbb{C} such that F is compatible with S and G is compatible with T . Then we have that S commutes with $T \Leftrightarrow$ We have for every $f \in C_0(F)$ and $g \in C_0(G)$ that $f(S)g(T) = g(T)f(S)$.*

Proof :

- Suppose that S commutes with T . Take $f \in C_0(F)$ and $g \in C_0(G)$. By lemma 6.2, there exist $k, h \in C_0(\mathbb{C})$ such that $f \subseteq h$ and $g \subseteq k$. This implies that $f(S)g(T) = k(S)h(T) = h(T)k(S) = g(T)f(S)$.
- Suppose that $f(S)g(T) = g(T)f(S)$ for every $f \in C_0(F)$ and $g \in C_0(G)$. Applying lemma 8.7 twice, this implies that $f(S)g(T) = g(T)f(S)$ for every $f \in C_b(F)$ and $g \in C_b(G)$.

So we get for every $h, k \in C_0(\mathbb{C})$ that $h(S)k(T) = (h_F)(S)(k_G)(T) = (k_G)(T)(h_F)(S) = k(T)h(T)$.

■

Using lemma 8.7, we get immediately the following results.

Result 12.3 Consider a Hilbert C^* -module E over a C^* -algebra A and let S, T be commuting normal elements in $\mathcal{R}(E)$. Let F, G be subsets of \mathbb{C} such that F is compatible with S and G is compatible with T . Then the following properties hold :

1. We have for $f \in C_b(F)$ and $g \in C_b(G)$ that $f(S)g(T) = g(T)f(S)$.
2. We have for $f \in C_b(F)$ and $g \in C(G)$ that $f(S)g(T) \subseteq g(T)f(S)$.
3. We have for $f \in C(F)$ and $g \in C_b(G)$ that $g(T)f(S) \subseteq f(S)g(T)$.

Combining result 12.2 with the Fuglede-Putnam theorem (theorem 8.8), we get immediately the following equivalent characterization of commuting normal operators.

Proposition 12.4 Consider a Hilbert C^* -module E over a C^* -algebra A and let S, T be normal elements in $\mathcal{R}(E)$ and let F be a subset of \mathbb{C} which is compatible with S . Then S commutes with $T \Leftrightarrow$ We have for every $f \in C_0(F)$ that $f(S)T \subseteq T f(S)$.

In the case that one of the two (or both) are bounded, we get the usual commutation conditions. The next result follows immediately from the previous one and the Fuglede-Putnam theorem.

Result 12.5 Consider a Hilbert C^* -module E over a C^* -algebra A , let S be a normal element in $\mathcal{L}(E)$ and T a normal element in $\mathcal{R}(E)$. Then S commutes with $T \Leftrightarrow ST \subseteq TS$.

Corollary 12.6 Consider a Hilbert C^* -module E over a C^* -algebra A , let S, T be normal elements in $\mathcal{L}(E)$. Then S commutes with $T \Leftrightarrow ST = TS$.

It is of course not necessary to look at all the elements of $C_0(F)$. A certain well behaved subset will suffice.

Result 12.7 Consider a Hilbert C^* -module E over a C^* -algebra A and let S, T be normal elements in $\mathcal{R}(E)$ and let F be a subset of \mathbb{C} which is compatible with S . Consider furthermore a bounded net $(e_i)_{i \in I}$ in $K(F)$ which converges strictly to 1.

Then S and T commute \Leftrightarrow We have for every $i \in I$ that $(S \cdot e_i(S))T \subseteq T(S \cdot e_i(S))$.

Proof : We already know that one implication is true. So we turn to the other one.

Therefore suppose that $(S \cdot e_i(S))T \subseteq T(S \cdot e_i(S))$ for every $i \in I$.

Take $f \in C_0(\mathbb{C})$.

By the Fuglede-Putnam theorem, we get that $(S \cdot e_i(S))f(T) = f(T)(S \cdot e_i(S))$ for every $i \in I$.

Choose $v \in D(S)$. Take $j \in I$.

We know that $e_j(S)f(T)v \in D(S)$ and that

$$S(e_j(S)f(T)v) = (S \cdot e_j(S))f(T)v = f(T)(S \cdot e_j(S))v = f(T)(e_j(S)S(v))$$

So we see that $(e_i(S)f(T)v)_{i \in I}$ converges to $f(T)v$ and that $(S(e_i(S)f(T)v))_{i \in I}$ converges to $f(T)S(v)$. Therefore the closedness of S implies that $f(T)v$ belongs to $D(S)$ and $S(f(T)v) = f(T)S(v)$.

So we have proven that $f(T)S \subseteq S f(T)$. ■

Combining the first statement of result 12.2 with proposition 6.19, the next useful result becomes easy to prove.

Corollary 12.8 *Consider a Hilbert C^* -module E over a C^* -algebra A and let S, T be commuting normal elements in $\mathcal{R}(E)$. Let F, G be subsets of \mathbb{C} such that F is compatible with S and G is compatible with T . Then we have for every $f \in C(F)$ and $g \in C(G)$ that $f(S)$ and $g(T)$ commute.*

Because z_T is a function of T and equation 1.19 of [13], we have of course the next corollary.

Corollary 12.9 *Consider a Hilbert C^* -module E over a C^* -algebra A and let S, T be normal elements in $\mathcal{R}(E)$. Then S commutes with $T \Leftrightarrow z_S z_T = z_T z_S$.*

The next result follows immediately from proposition 6.17.

Result 12.10 *Consider a Hilbert C^* -module E over a C^* -algebra A . Let B be a C^* -algebra A and π a non-degenerate $*$ -homomorphism from B into $\mathcal{L}(E)$. Suppose that S and T are two normal elements affiliated with B such that S and T commute. Then $\pi(S)$ and $\pi(T)$ commute.*

As could be expected, the functional calculi of two commuting normal operators give a functional calculus on $\mathbb{C} \times \mathbb{C}$. This is essentially the content of proposition 12.12.

First, we state some unicity result.

Result 12.11 *Consider a Hilbert C^* -module E over a C^* -algebra A and let S, T be commuting normal elements in $\mathcal{R}(E)$ and let F, G be almost closed subsets of \mathbb{C} . Define p as the projection of $F \times G$ on F and q as the projection of $F \times G$ on G . Consider non-degenerate $*$ -homomorphism π, θ from $C_0(F \times G)$ into $\mathcal{L}(E)$. If $\pi(p) = \theta(p)$ and $\pi(q) = \theta(q)$, then $\pi = \theta$.*

Proof : Choose $f \in C_0(F)$ and $g \in C_0(G)$.

By lemma 6.2, there exists $h \in C_0(\mathbb{C})$ such that $f \subseteq h$. Using proposition 3.9, we see that $\pi(f \otimes 1) = \pi(h \circ p) = \pi(h(p)) = h(\pi(p))$ and similarly, $\theta(f \otimes 1) = h(\theta(p))$. Hence, $\pi(f \otimes 1) = \theta(f \otimes 1)$.

One proves in the same way that $\pi(1 \otimes g) = \theta(1 \otimes g)$. Consequently, $\pi(f \otimes g) = \theta(f \otimes g)$. ■

Proposition 12.12 *Consider a Hilbert C^* -module E over a C^* -algebra A and let S, T be commuting normal elements in $\mathcal{R}(E)$ and let F, G be subsets of \mathbb{C} such that F is compatible with S and G is compatible with T . Define p as the projection of $F \times G$ on F and q as the projection of $F \times G$ on G . Then there exists a unique non-degenerate $*$ -homomorphism π from $C_0(F \times G)$ into $\mathcal{L}(E)$ such that $\pi(p) = S$ and $\pi(q) = T$. We call π the functional calculus of S, T on $F \times G$. Furthermore,*

- *We have for every $f \in C(F)$ that $\pi(f \otimes 1) = f(S)$*
- *We have for every $g \in C(G)$ that $\pi(1 \otimes g) = g(T)$*

Proof : Call θ the functional calculus of S on F and ρ the functional calculus of T on G . Because S and T commute, we have by result 12.2 that θ and ρ have commuting ranges. Hence, the universal property of the maximal tensor product implies the existence of a unique $*$ -homomorphism π from $C_0(F \times G)$ such that $\pi(f \otimes g) = \theta(f)\rho(g) = \rho(g)\theta(f)$ for $f \in C_0(F)$ and $g \in C_0(G)$.

Take $h \in C(F)$

Choose an approximate unit $(e_i)_{i \in I}$ for $C_0(F)$ in $K(F)$ and an approximate unit $(u_j)_{j \in J}$ for $C_0(G)$ in $K(G)$. Then $(e_i \otimes u_j)_{(i,j) \in I \times J}$ is an approximate unit for $C_0(F \times G)$ in $K(F \times G)$.

Hence, $(\pi(e_i \otimes u_j))_{(i,j) \in I \times J}$ is a bounded net which converges strongly to 1.

Fix $i \in I, j \in J$.

Because $e_i \otimes u_j$ belongs to $K(F \otimes G)$, lemma 4.3 implies that $\pi(e_i \otimes u_j) E \subseteq D(\pi(h \otimes 1))$ and

$$\pi(e_i \otimes u_j) \pi(h \otimes 1) \subseteq \pi(h \otimes 1) \pi(e_i \otimes u_j) \quad (1)$$

We have also that $\pi(e_i \otimes u_j) = e_i(S) u_j(T)$. This implies that $\pi(e_i \otimes u_j) E \subseteq D(h(S))$. Because S and T commute, we have moreover that

$$\pi(e_i \otimes u_j) h(S) \subseteq h(S) \pi(e_i \otimes u_j) \quad (2)$$

We see also that

$$\begin{aligned} \pi(h \otimes 1) \pi(e_i \otimes u_j) &= \pi((h \otimes 1)(e_i \otimes u_j)) = \pi((he_i) \otimes u_j) \\ &= (he_i)(S) u_j(T) = h(S) e_i(S) u_j(T) = h(S) \pi(e_i \otimes u_j) \end{aligned} \quad (3)$$

Combining these three results and using the closedness of both $\pi(h \otimes 1)$ and $h(S)$, we get easily that $\pi(h \otimes 1) = h(S)$.

So we get in particular that $\pi(p) = \pi(\iota_F \otimes 1) = \iota_F(S) = S$.

The results concerning elements in $C(G)$ are proven in the same way. ■

Notation 12.13 Consider a Hilbert C^* -module E over a C^* -algebra A and let S, T be commuting normal elements in $\mathcal{R}(E)$ and let F, G be subsets of \mathbb{C} such that F is compatible with S and G is compatible with T . Call π the functional calculus of S, T on $F \times G$. For every $h \in C(F \times G)$, we define $h(S, T) = \pi(h)$, so $h(S, T)$ is a normal element in $\mathcal{R}(E)$.

Result 12.14 Consider a Hilbert C^* -module E over a C^* -algebra A and let S, T be commuting normal elements in $\mathcal{R}(E)$ and let F, G be subsets of \mathbb{C} such that F is compatible with S and G is compatible with T . Then we have the following properties :

- We have for every $f \in C(F)$ that $(f \otimes 1)(S, T) = f(S)$.
- We have for every $g \in C(G)$ that $(1 \otimes g)(S, T) = g(T)$.

Corollary 12.15 Consider a Hilbert C^* -module E over a C^* -algebra A and let S, T be commuting normal elements in $\mathcal{R}(E)$ and let F, G be subsets of \mathbb{C} such that F is compatible with S and G is compatible with T . Then we have for every $f \in C_b(F)$ and $g \in C_b(G)$ that $(f \otimes g)(S, T) = f(S)g(T) = g(T)f(S)$.

Of course, restricting functions does not change the value.

Result 12.16 Consider a Hilbert C^* -module E over a C^* -algebra A and let S, T be commuting normal elements in $\mathcal{R}(E)$ and let F, G, H, K be subsets of \mathbb{C} such that F, H are compatible with S , G, K are compatible with T such that $F \subseteq H$ and $G \subseteq K$. Then we have for $h \in C(H \times K)$ that $h(S, T) = h_{F \times G}(S, T)$.

The proof of this result is the same as the proof of lemma 6.7.

As usual, the next result turns out to be very useful. It is an immediate consequence of proposition 6.17 and result 6.18.

Result 12.17 Consider a Hilbert C^* -module E over a C^* -algebra A and let S, T be commuting normal elements in $\mathcal{R}(E)$ and let F, G be subsets of \mathbb{C} such that F is compatible with S and G is compatible with T . Consider $k \in C(F \times G)$ and consider an almost closed subset H of \mathbb{C} such that $k(F \otimes G) \subseteq H$. Then we have the following properties :

- The set H is compatible with $k(S, T)$.
- We have for every $h \in C(H)$ that $h(k(S, T)) = (h \circ k)(S, T)$.

This functional calculus for the pair S, T allows us to define the product and the sum of two commuting normal operators. Call p the projection of $\mathbb{C} \times \mathbb{C}$ on the first factor and q the projection of $\mathbb{C} \times \mathbb{C}$ on the second factor. Then $p(S, T) = S$ and $q(S, T) = T$. Hence, looking at $(p + q)(S, T)$ and $(pq)(S, T)$ and referring to results 4.10 and 4.5, the following definition is justified.

Definition 12.18 Consider a Hilbert C^* -module E over a C^* -algebra A and let S, T be commuting normal elements in $\mathcal{R}(E)$.

- The element ST is closable and we define $S \cdot T = \overline{ST}$. Then $S \cdot T$ is a normal element in $\mathcal{R}(E)$.
- The element $S + T$ is closable and we define $S \dot{+} T = \overline{S + T}$. Then $S \dot{+} T$ is a normal element in $\mathcal{R}(E)$.

Referring to results 4.10 and 4.5 once more, we get immediately the following result.

Corollary 12.19 Consider a Hilbert C^* -module E over a C^* -algebra A and let S, T be commuting normal elements in $\mathcal{R}(E)$ and let F, G be subsets of \mathbb{C} such that F is compatible with S and G is compatible with T . Consider $f \in C(F)$ and $g \in C(G)$. Then

- $f(S) \cdot g(T) = g(T) \cdot f(S) = (f \otimes g)(S, T)$
- $f(S) \dot{+} g(T) = g(T) \dot{+} f(S) = (f \otimes 1 + 1 \otimes g)(S, T)$.

So we have in particular that $S \dot{+} T = T \dot{+} S$ and $S \cdot T = T \cdot S$.

We will not pursue this matter any further except for the next two results which will be useful in the next section.

Result 12.20 Consider a Hilbert C^* -module E over a C^* -algebra A and let S, T be commuting normal elements in $\mathcal{R}(E)$.

- If S and T are invertible, then $S \cdot T$ is invertible.
- If S and T are selfadjoint, then $S \cdot T$ is selfadjoint.
- If S and T are positive, then $S \cdot T$ is positive.
- If S and T are strictly positive, then $S \cdot T$ is strictly positive.

Proposition 12.21 Consider a Hilbert C^* -module E over a C^* -algebra A and let S, T be commuting normal elements in $\mathcal{R}(E)$.

1. We have that $(S \cdot T)^n = S^n \cdot T^n$ for $n \in \mathbb{N}$.
2. If S and T are invertible, then $(S \cdot T)^n = S^n \cdot T^n$ for $n \in \mathbb{Z}$.

3. If S and T are positive, then $(S \cdot T)^r = S^r \cdot T^r$ for $r \in \mathbb{R}^+$.
4. If S and T are strictly positive, then $(S \cdot T)^z = S^z \cdot T^z$ for $z \in \mathbb{C}$.

Remark 12.22 As an example, we will prove the first statement of the first result and the second statement of the second one. The proofs of the other ones are completely similar.

So take commuting invertible normal elements S, T in $\mathcal{R}(E)$. Then \mathbb{C}_0 is compatible with S and T . By corollary 12.19, we get that

$$S \cdot T = \iota_{\mathbb{C}_0}(S) \cdot \iota_{\mathbb{C}_0}(T) = (\iota_{\mathbb{C}_0} \otimes \iota_{\mathbb{C}_0})(S, T)$$

It is clear that $\iota_{\mathbb{C}_0} \otimes \iota_{\mathbb{C}_0}$ is invertible in $C(\mathbb{C}_0 \times \mathbb{C}_0)$. This implies that $S \cdot T$ is invertible.

Choose $n \in \mathbb{Z}$ and define the function h from \mathbb{C}_0 into \mathbb{C}_0 such that $h(c) = c^n$ for $c \in \mathbb{C}_0$. Then result 12.17 implies that

$$\begin{aligned} (S \cdot T)^n &= h(S \cdot T) = h((\iota_{\mathbb{C}_0} \otimes \iota_{\mathbb{C}_0})(S, T)) = (h \circ (\iota_{\mathbb{C}_0} \otimes \iota_{\mathbb{C}_0}))(S, T) \\ &= (h \otimes h)(S, T) = h(S) \cdot h(T) = S^n \cdot T^n \end{aligned}$$

We end this section with the familiar result that the commuting relation involving strictly positive elements can be stated in terms of the unitary groups they generate.

In [5], we prove the following Hilbert C^* -module version of a well known Hilbert space result (see e.g. [10]).

For any $z \in \mathbb{C}$, we use the notation $S(z) = \{ y \in \mathbb{C} \mid \text{Im } y \in [0, \text{Im } z] \}$.

Proposition 12.23 Consider a Hilbert C^* -module E over a C^* -algebra A and let S be a strictly positive element in $\mathcal{R}(E)$. Let z be a complex number and $v \in E$.

Then we have that v belongs to $D(S^{iz}) \Leftrightarrow$ There exists a function f from $S(z)$ into E such that f is continuous on $S(z)$, f is analytic on $S(z)^0$ and $f(t) = S^{it}v$ for $t \in \mathbb{R}$.

If such a function f exists, then $S^{iz}v = f(z)$.

This allows us to prove the following commutation criteria.

Result 12.24 Consider a Hilbert C^* -module E over a C^* -algebra A and let S be a strictly positive element in $\mathcal{R}(E)$. Let T a normal element in $\mathcal{R}(E)$.

Then S commutes with $T \Leftrightarrow$ We have for every $t \in \mathbb{R}$ and that $S^{it}T \subseteq TS^{it}$.

Proof : Suppose that $S^{it}T \subseteq TS^{it}$ for every $t \in \mathbb{R}$.

Choose $g \in C_0(G)$. Then Fuglede-Putnam implies that $S^{it}g(T) = g(T)S^{it}$. The previous proposition implies (with a small effort) that $g(T)S \subseteq Sg(T)$. Hence, S and T commute by proposition 12.4. ■

Applying the previous result, we get the following one.

Corollary 12.25 Consider a Hilbert C^* -module E over a C^* -algebra A and let S, T be strictly positive element in $\mathcal{R}(E)$. Then S and T commute \Leftrightarrow We have for every $s, t \in \mathbb{R}$ that $S^{is}T^{it} = T^{it}S^{is}$.

13 Tensor products of Hilbert C^* -modules

In this section, we will look at some basic properties of regular operators on the tensor products of Hilbert C^* -modules. We will always work with the minimal tensor product between C^* -algebras.

In chapter 4 of [7], the tensor product of two Hilbert C^* -modules was defined in the following way.

Definition 13.1 *Consider a Hilbert C^* -module E over a C^* -algebra A and a Hilbert C^* -module F over a C^* -algebra B . Then $E \otimes F$ is defined to be a Hilbert C^* -module over $A \otimes B$ such that*

- *The set $E \odot F$ is a dense subspace of $E \otimes F$.*
- *We have for every $v_1, v_2 \in E$ and $w_1, w_2 \in F$ that $\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \langle v_1, v_2 \rangle \otimes \langle w_1, w_2 \rangle$.*

In the same chapter of [7], the tensor product of two adjointable operators was defined in the following way.

Proposition 13.2 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and Hilbert C^* -modules G, H over a C^* -algebra B . Let $S \in \mathcal{L}(E, F)$ and $T \in \mathcal{L}(G, H)$. Then there exists a unique $R \in \mathcal{L}(E \otimes F, G \otimes H)$ such that $R(v \otimes w) = S(v) \otimes T(w)$ for every $v \in E$ and $w \in G$. We have moreover that $\|R\| = \|S\| \|T\|$.*

Lemma 13.3 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and Hilbert C^* -modules G, H over a C^* -algebra B . By the universality property of the algebraic tensor product, there exists a unique linear map π from $\mathcal{L}(E, F) \odot \mathcal{L}(G, H)$ into $\mathcal{L}(E \otimes G, F \otimes H)$ such that $\pi(S \otimes T)(v \otimes w) = S(v) \otimes T(w)$ for $S \in \mathcal{L}(E, F)$, $T \in \mathcal{L}(G, H)$, $v \in E$, $w \in G$. Then π is injective.*

Proof : Take $x \in \mathcal{L}(E, F) \odot \mathcal{L}(G, H)$ such that $\pi(x) = 0$. Then there exists $S_1, \dots, S_n \in \mathcal{L}(E, F)$ and $T_1, \dots, T_n \in \mathcal{L}(G, H)$ such that $x = \sum_{i=1}^n S_i \otimes T_i$ and such that S_1, \dots, S_n are linearly independent. Choose $v \in G$, $w \in H$. Take $\omega \in A^*$. Fix $p \in E$, $q \in F$. We have that

$$\sum_{i=1}^n \langle S_i(p), q \rangle \otimes \langle T_i(v), w \rangle = \langle \pi(x)(p \otimes v), q \otimes w \rangle = 0.$$

Applying $\omega \otimes \iota$ to this equation gives us that

$$0 = \sum_{i=1}^n \langle S_i(p), q \rangle \omega(\langle T_i(v), w \rangle) = \langle [\sum_{i=1}^n \omega(\langle T_i(v), w \rangle) S_i](p), q \rangle$$

Hence, we see that $\sum_{i=1}^n \omega(\langle T_i(v), w \rangle) S_i = 0$. By the linear independence of S_1, \dots, S_n , we get for every $i \in \{1, \dots, n\}$ that $\omega(\langle T_i(v), w \rangle) = 0$.

Consequently, $T_1 = \dots = T_n = 0$. So $x = 0$. ■

From now on, we will forget about the mapping π and consider $\mathcal{L}(E, F) \odot \mathcal{L}(G, H)$ as a subspace of $\mathcal{L}(E \otimes G, F \otimes H)$. So we get the following result.

Result 13.4 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and Hilbert C^* -modules G, H over a C^* -algebra B . Let $S \in \mathcal{L}(E, F)$ and $T \in \mathcal{L}(G, H)$. Then $S \otimes T$ is the unique element in $\mathcal{L}(E \otimes G, F \otimes H)$ such that $(S \otimes T)(v \otimes w) = S(v) \otimes T(w)$ for $v \in E$ and $w \in G$. We have moreover that $\|S \otimes T\| = \|S\| \|T\|$.*

As in the case of Hilbert spaces, the minimal tensor product norm of an element in $\mathcal{L}(E) \odot \mathcal{L}(F)$ is equal to the norm of it as an operator on $E \otimes F$.

Proposition 13.5 *Consider a Hilbert C^* -module E over a C^* -algebra A and a Hilbert C^* -module F over a C^* -algebra B . Then we have for every $x \in \mathcal{L}(E) \odot \mathcal{L}(F)$ that $\|x\|_{\min} = \|x\|$ (where the last norm is the norm of x as an element in $\mathcal{L}(E \otimes F)$).*

Proof : If we restrict the norm on $\mathcal{L}(E \otimes F)$ to $\mathcal{L}(E) \odot \mathcal{L}(F)$, we get a C^* -norm on $\mathcal{L}(E) \odot \mathcal{L}(F)$. This implies immediately that $\|x\|_{\min} \leq \|x\|$. We will turn to the other inequality.

Take $\varepsilon > 0$

Use the notations of section 11. Define the set $K = \{\rho \otimes \theta \mid \rho \in A_+^*, \theta \in B_+^*\}$. Because we work with the minimal tensor product on $A \otimes B$, the set K is separating for $A \otimes B$.

Define the mapping η from $\mathcal{L}(E \otimes F)$ into $\prod_{\omega \in K} \mathcal{B}((E \otimes F)_\omega)$ such that $(\eta(y))_\omega = y_\omega$ for every $y \in \mathcal{L}(E \otimes F)$ and $\omega \in K$. Then lemma 11.15 implies that η is an injective $*$ -homomorphism and consequently isometric. Hence,

$$\|x\| = \|\eta(x)\| = \sup \{ \|x_\omega\| \mid \omega \in K \}$$

This implies the existence of $\rho \in A_+^*$ and $\theta \in B_+^*$ such that $\|x_{\omega \otimes \theta}\| \geq \|x\| - \varepsilon$.

Define the mappings $\eta_\rho : \mathcal{L}(E) \rightarrow \mathcal{B}(E_\rho) : a \mapsto a_\rho$ and $\eta_\theta : \mathcal{L}(F) \rightarrow \mathcal{B}(F_\theta) : b \mapsto b_\theta$. Then η_ρ and η_θ are $*$ -homomorphisms. Hence we get the inequality $\|x\|_{\min} \geq \|(\eta_\rho \otimes \eta_\theta)(x)\|$.

It is easy to check that $\langle \overline{v_1} \otimes \overline{w_1}, \overline{v_2} \otimes \overline{w_2} \rangle = \langle \overline{v_1 \otimes w_1}, \overline{v_2 \otimes w_2} \rangle$ for $v_1, v_2 \in E$ and $w_1, w_2 \in F$. This implies the existence of a unitary operator U from $E_\rho \otimes F_\theta$ to $(E \otimes F)_{\rho \otimes \theta}$ such that $U(\overline{v} \otimes \overline{w}) = \overline{v \otimes w}$ for $v \in E$ and $w \in F$.

Now it is straightforward to check that $(\eta_\rho \otimes \eta_\theta)(x) = U^* x_{\omega \otimes \theta} U$ which implies that

$$\|x\|_{\min} \geq \|(\eta_\rho \otimes \eta_\theta)(x)\| = \|x_{\omega \otimes \theta}\| \geq \|x\| - \varepsilon$$

From this all, we can conclude that $\|x\|_{\min} \geq \|x\|$. Hence, $\|x\|_{\min} = \|x\|$. ■

So from now on, we can consider $\mathcal{L}(E) \otimes \mathcal{L}(F)$ as the closure of $\mathcal{L}(E) \odot \mathcal{L}(F)$ in $\mathcal{L}(E \otimes F)$.

It is even possible to define the tensor product of two regular operators (see theorem 6.1 of [8] and chapter 10 of [7]).

Definition 13.6 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and Hilbert C^* -modules G, H over a C^* -algebra B . Consider moreover elements $S \in \mathcal{R}(E, F)$ and $T \in \mathcal{R}(G, H)$. Then $S \odot T$ is closable and we define $S \otimes T$ to be the closure of $S \odot T$.*

It is not difficult to check that $C \odot D$ is a core for $S \otimes T$ if C is a core for S and D is a core for T .

Proposition 13.7 *Consider Hilbert C^* -modules E, F over a C^* -algebra A and Hilbert C^* -modules G, H over a C^* -algebra B . Consider moreover elements $S \in \mathcal{R}(E, F)$ and $T \in \mathcal{R}(G, H)$. Then $S \otimes T$ belongs to $\mathcal{R}(E \otimes G, F \otimes H)$ and $(S \otimes T)^* = S^* \otimes T^*$.*

The tensor product of normal (resp. selfadjoint, positive) elements will again be normal (resp. selfadjoint, positive):

Result 13.8 *Consider a Hilbert C^* -module E over a C^* -algebra A and a Hilbert C^* -module F over a C^* -algebra B . Consider moreover elements $S \in \mathcal{R}(E)$ and $T \in \mathcal{R}(F)$. Then we have the following properties.*

- If S and T are normal, then $S \otimes T$ is normal.
- If S and T are selfadjoint, then $S \otimes T$ is selfadjoint.
- If S and T are positive, then $S \otimes T$ is positive.
- If S and T are strictly positive, then $S \otimes T$ is strictly positive.

Proof :

- The case for selfadjoint elements follows easily by the previous proposition.
- Suppose that S and T are normal. By polarisation, we have that $\langle S(v), S(w) \rangle = \langle S^*(v), S^*(w) \rangle$ for every $v, w \in D(S)$. A similar remark applies to T . Using these equalities, it is straightforward to check that $D(S \odot T) = D(S^* \odot T^*)$ and that

$$\langle (S \odot T)(u), (S \odot T)(u) \rangle = \langle (S^* \odot T^*)(u), (S^* \odot T^*)(u) \rangle$$

for every $u \in D(S \odot T)$.

This implies easily that $D(S \otimes T) = D(S^* \otimes T^*)$ and that

$$\langle (S \otimes T)(u), (S \otimes T)(u) \rangle = \langle (S^* \otimes T^*)(u), (S^* \otimes T^*)(u) \rangle$$

for every $u \in D(S \otimes T)$. So $S \otimes T$ is normal by definition.

- Suppose that S and T are positive. We know already that $S \otimes T$ is selfadjoint. Choose $v_1, \dots, v_n \in D(S)$ and $w_1, \dots, w_n \in D(T)$. Define $M \in M_n(A)$ and $N \in M_n(B)$ such that $M_{ij} = \langle S(v_i), v_j \rangle$ and $N_{ij} = \langle T(w_i), w_j \rangle$ for every $i, j \in \{1, \dots, n\}$. Using lemma 3.2 of [12] and the positivity of S , it follows readily that the matrix M is positive. Similarly we find that N is positive. Using lemma 4.3 of [7], we find that $\sum_{i,j=1}^n M_{ij} \otimes N_{ij}$ is positive. But we have also that

$$\sum_{i,j=1}^n M_{ij} \otimes N_{ij} = \langle (S \otimes T) \left(\sum_{i=1}^n v_i \otimes w_i \right), \left(\sum_{j=1}^n v_j \otimes w_j \right) \rangle$$

which implies that the right hand side is positive.

So we have proven that $\langle (S \otimes T)(u), u \rangle$ is positive for every $u \in D(S) \odot D(T)$. From this we get that $\langle (S \otimes T)(u), u \rangle$ is positive for every $u \in D(S \otimes T)$. So $S \otimes T$ is positive.

- The result concerning strict positivity follows now easily.

■

Result 13.9 Consider Hilbert C^* -modules E, F over a C^* -algebra A and Hilbert C^* -modules G, H over a C^* -algebra B . Consider moreover elements $S \in \mathcal{R}(E, F)$ and $T \in \mathcal{R}(G, H)$.

- If S and T are invertible, then $S \otimes T$ is invertible and $(S \otimes T)^{-1} = S^{-1} \otimes T^{-1}$.
- If S and T are adjointable invertible, then $S \otimes T$ is adjointable invertible.

Proof : Suppose that S and T are invertible. Using proposition 5.4, it is easy to see that $S \otimes T$ and $S^* \otimes T^*$ are invertible.

We have moreover that $S^{-1} \odot T^{-1} = (S \odot T)^{-1} \subseteq (S \otimes T)^{-1}$ which implies that $S^{-1} \otimes T^{-1} \subseteq (S \otimes T)^{-1}$. In the same way, it follows that $(S^*)^{-1} \otimes (T^*)^{-1} \subseteq (S^* \otimes T^*)^{-1}$, which implies that $(S^{-1} \otimes T^{-1})^* \subseteq ((S \otimes T)^{-1})^*$. Hence, $(S \otimes T)^{-1} \subseteq S^{-1} \otimes T^{-1}$. Consequently, $S^{-1} \otimes T^{-1} = (S \otimes T)^{-1}$.

The second statement follows immediately from the first. \blacksquare

Result 13.10 Consider Hilbert C^* -modules E, F over a C^* -algebra A and Hilbert C^* -modules G, H over a C^* -algebra B . Consider moreover elements $S \in \mathcal{R}(E, F)$ and $T \in \mathcal{R}(G, H)$. Then we have the equalities

$$(S \otimes T)^*(S \otimes T) = S^*S \otimes T^*T \quad (S \otimes T)(S \otimes T)^* = SS^* \otimes TT^*$$

Proof : We have that

$$S^*S \odot T^*T \subseteq (S^* \odot T^*)(S \odot T) \subseteq (S^* \otimes T^*)(S \otimes T) = (S \otimes T)^*(S \otimes T).$$

This implies that $S^*S \otimes T^*T \subseteq (S \otimes T)^*(S \otimes T)$. Because these two elements are selfadjoint, this inclusion is an equality. \blacksquare

Now we have our obligatory non-degenerate $*$ -homomorphism result.

Proposition 13.11 Consider a Hilbert C^* -module E over a C^* -algebra A and a Hilbert C^* -module F over a C^* -algebra B . Let C and D be C^* -algebras, π a non-degenerate $*$ -homomorphism from C into $\mathcal{L}(E)$ and θ a non-degenerate $*$ -homomorphism from D into $\mathcal{L}(F)$. Then we have for every $S \in \mathcal{R}(C)$ and $T \in \mathcal{R}(D)$ that $(\pi \otimes \theta)(S \otimes T)$.

Proof : Because $D(S) \odot D(T)$ is a core for $S \otimes T$ and $E \odot F$ is dense in $E \otimes F$, the remark after theorem 1.9 implies that $(\pi \otimes \theta)(D(S) \odot D(T))(E \odot F)$ is a core for $(\pi \otimes \theta)(S \otimes T)$. Hence, $\pi(D(S))E \odot \theta(D(T))F$ is a core for $(\pi \otimes \theta)(S \otimes T)$.

Because $\pi(D(S))E$ is a core for $\pi(S)$ and $\theta(D(T))F$ is a core for $\theta(T)$, the remark after definition 13.6 gives us that $\pi(D(S))E \odot \theta(D(T))F$ is also a core for $\pi(S) \otimes \theta(T)$.

It is easy to see that

$$(\pi \otimes \theta)(S \otimes T) (\pi(c)v \otimes \theta(d)w) = (\pi(S) \otimes \theta(T)) (\pi(c)v \otimes \theta(d)w)$$

for $c \in D(S)$, $d \in D(T)$, $v \in E$ and $w \in F$. Consequently, $(\pi \otimes \theta)(S \otimes T) = \pi(S) \otimes \theta(T)$. \blacksquare

Lemma 13.12 Consider a Hilbert C^* -module E over a C^* -algebra A and a Hilbert C^* -module F over a C^* -algebra B . Define the injective non-degenerate $*$ -homomorphism π from $\mathcal{K}(E)$ into $\mathcal{L}(E \otimes F)$ such that $\pi(x) = x \otimes 1$ for every $x \in \mathcal{K}(E)$. Then we have for every $S \in \mathcal{R}(\mathcal{K}(E))$ that $\pi(S) = \tilde{S} \otimes 1$.

Proof : We know that $D(S)E$ is a core for \tilde{S} (result 10.3). This implies that $D(S)E \odot F$ is a core for $\tilde{S} \otimes 1$. On the other hand, we know also that $\pi(D(S))(E \odot F)$ is a core for $\pi(S)$. This implies that $D(S)E \odot F$ is also a core for $\pi(D(S))$.

We have moreover for every $x \in D(S)$ and $v \in E$, $w \in F$ that

$$\pi(S)(xv \otimes w) = \pi(S)(\pi(x)(v \otimes w)) = \pi(S(x))(v \otimes w) = S(x)v \otimes w = \tilde{S}(xv) \otimes w = (\tilde{S} \otimes 1)(xv \otimes w)$$

which implies that $\pi(S) = \tilde{S} \otimes 1$. \blacksquare

By corollary 10.11 and proposition 6.17, this implies immediately the following result.

Corollary 13.13 Consider a Hilbert C^* -module E over a C^* -algebra A and a Hilbert C^* -module F over a C^* -algebra B . Let S be a normal element in $\mathcal{R}(E)$ and G a subset of \mathbb{C} which is compatible with S . Then we have the following properties :

- The set G is compatible with $S \otimes 1$.
- We have for every $f \in C(G)$ that $f(S \otimes 1) = f(S) \otimes 1$.

We have of course similar results for elements of the form $1 \otimes T$.

Result 13.14 Consider a Hilbert C^* -module E over a C^* -algebra A and a Hilbert C^* -module F over a C^* -algebra B . Let S be a normal element in $\mathcal{R}(E)$ and T a normal element in $\mathcal{R}(F)$. Then $S \otimes 1$ and $1 \otimes T$ commute and $(S \otimes 1) \cdot (1 \otimes T) = S \otimes T$.

Proof : The fact that $S \otimes 1$ and $1 \otimes T$ commute follows immediately from the previous result and the remark after it. We have moreover that

$$S \odot T \subseteq (S \odot 1)(1 \odot T) \subseteq (S \otimes 1)(1 \otimes T) = (S \otimes 1) \cdot (1 \otimes T)$$

which implies that $S \otimes T \subseteq (S \otimes 1) \cdot (1 \otimes T)$. Because both are normal, this inclusion is an equality. ■

So we can also look at functional calculi for the pair S, T .

Looking at the previous result, corollary 13.13 and corollary 12.19, we see that the following holds.

Result 13.15 Consider a Hilbert C^* -module E over a C^* -algebra A and a Hilbert C^* -module over a C^* -algebra B . Let S be a normal element in $\mathcal{R}(E)$ and T a normal element in $\mathcal{R}(F)$. Consider subsets F, G of \mathbb{C} such that F is compatible with S and G is compatible with T . Then we have for every $f \in C(F)$ and $g \in C(G)$ that $(f \otimes g)(S \otimes 1, 1 \otimes T) = f(S) \otimes g(T)$.

The next result is an easy consequence of result 13.14, corollary 13.13 and proposition 12.21.

Proposition 13.16 Consider a Hilbert C^* -module E over a C^* -algebra A and a Hilbert C^* -module over a C^* -algebra B . Let S be a normal element in $\mathcal{R}(E)$ and T a normal element in $\mathcal{R}(F)$.

1. We have that $(S \otimes T)^n = S^n \otimes T^n$ for $n \in \mathbb{N} \cup \{0\}$.
2. If S and T are invertible, then $(S \otimes T)^n = S^n \otimes T^n$ for $n \in \mathbb{Z}$.
3. If S and T are positive, then $(S \otimes T)^r = S^r \otimes T^r$ for $r \in \mathbb{R}^+$.
4. If S and T are strictly positive, then $(S \otimes T)^z = S^z \otimes T^z$ for $z \in \mathbb{C}$.

Corollary 13.17 Consider Hilbert C^* -modules E, F over a C^* -algebra A and Hilbert C^* -modules G, H over a C^* -algebra B . Consider moreover elements $S \in \mathcal{R}(E, F)$ and $T \in \mathcal{R}(G, H)$. Then we have that $|S \otimes T| = |S| \otimes |T|$.

We end this section with a problem connected with remark 7.24. It is not clear to me under which conditions it is possible to cut out bigger pieces out of the spectrum and retain a functional calculus on this set.

If you have a continuous function on a locally compact space, you can easily cut out closed sets which do not meet the image of this function.

I wonder in particular if the following is true.

Question 13.18 Consider a Hilbert C^* -module E over a C^* -algebra A and let T be a normal element in $\mathcal{R}(E)$. Consider moreover a closed subset K of $\sigma(T)$ such that the closure of $T \otimes 1 - 1 \otimes \iota_K$ is invertible in $\mathcal{R}(E \otimes C_0(K))$.

Does there exist in this case a non-degenerate $*$ -homomorphism π from $C_0(\sigma(T) \setminus K)$ into $\mathcal{L}(E)$ such that $\pi(\iota_{\sigma(T) \setminus K}) = T$.

It is not hard to check that the answer is affirmative in the commutative case and the case where K is finite (by proposition 6.5).

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